Alpha Found Video 0

Overview

Welcome to the Alpha Found video series overview.

The goal of this series of videos is to introduce and defend a new model for stock market dynamics. The new model predicts, as both a mathematical expectation and economic expectation, that low volatility portfolios will outperform high volatility portfolios on a risk adjusted basis. That is, the long observed positive alpha of low beta portfolios is not an economic anomaly that needs to be explained, but an expectation. It would be an anomaly if it didn't exist.

This video, numbered 0, is intended for investment professionals. Students who have covered the Capital Asset Pricing Model in class should start with the next video, numbered 1.

Carl Sagan has said, "extraordinary claims require extraordinary evidence." On the other hand, George Box said, "All models are wrong, but some are useful."

The mathematics of the model is internally consistent and predicts ex-ante the high alphas of low volatility portfolios. At a minimum, the model is useful.

The nature of this, and really any new model, is such that it cannot be understood in just a few minutes. There are over 200 numbered mathematical expressions, many of which are trivial, but many are not trivial. They have to be derived, implemented, and absorbed. If it is any consolation, however long it may take you to understand the model, you can be sure it took me longer to derive.

The pace of the videos is too fast for the content to be understood watching the first time. I recommend you stop the videos often, and rewatch. There are no sponsors for the series who will benefit.

Initially there were 28 videos, each of which has been reloaded multiple times to correct for typos and one major renumbering of the mathematical expressions. Later, it seemed that more videos were needed.

I will add videos as time permits, but there is no expectation that there will be regular updates.

The first eight videos after this one review the history and current state of capital market theory. They also describe what skills are needed if the model is to be understood. They are not intended to be comprehensive. For those who are comfortable with MPT, CAPM, Sharpe ratios, and multifactor models, you could begin with the nineth video called Setting The Table, Part 1. As a practical matter, I don't think it's possible to understand what the model means if you skip videos after the first eight.

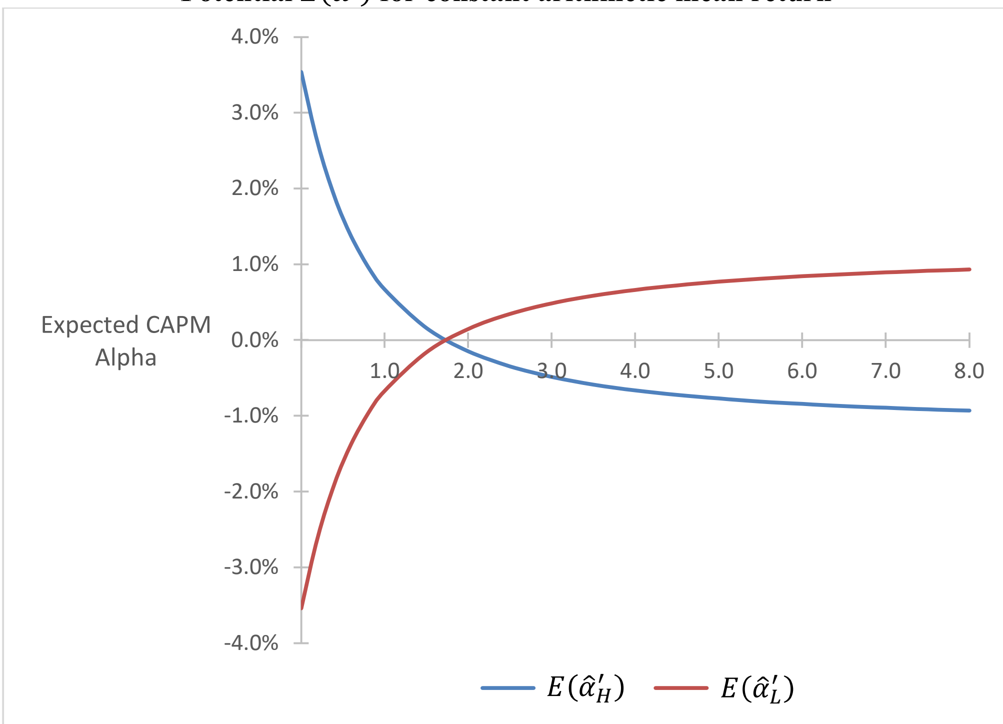
Videos nine and ten introduce a few thoughts that will be used to motivate and derive the new model.

Videos eleven through twenty have a simplifying assumption that gets removed in the twenty-first video. The simplifying assumption makes it much easier to understand the model and its implications without getting lost in the mathematics. When the assumption is used, it's called the "Naïve Model." When the assumption is lifted, it's called the "Full Model." There are other simplifying assumptions; however, those assumptions pose no threat to the overall usefulness of the models.

Videos twenty-one through twenty-eight explain the full model. Anything that comes after the twenty-eighth video is largely a mystery as I have not made most of those videos as of yet.

Let me take a moment to provide a hook. If the world is divided evenly between a high beta portfolio and a low beta portfolio, then in the new model there is a range of potential expected alphas for the same arithmetic return, as shown in this graph, where the red curve is the expected alpha for the low beta portfolio, and the blue curve has the expected alpha for the high beta portfolio.

Figure 0.1.
Potential $E(\hat{\alpha}')$ for constant arithmetic mean return

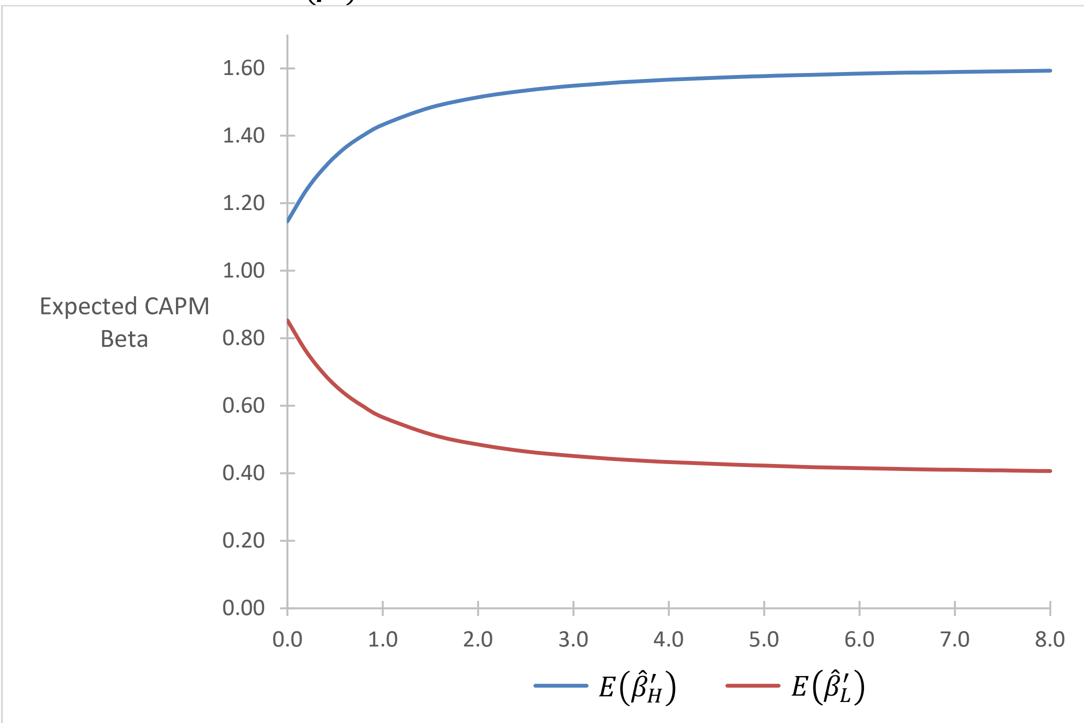


This is not the same as potential realized alphas, but expected alphas in the sense of mathematical expectation, depending on conditions. Note that this means expected alphas fall on a curve. Expected alpha does not take on a point value until certain market parameters are set.

I have left the horizontal axis unlabeled for the time being because I don't want to get too far ahead of things.

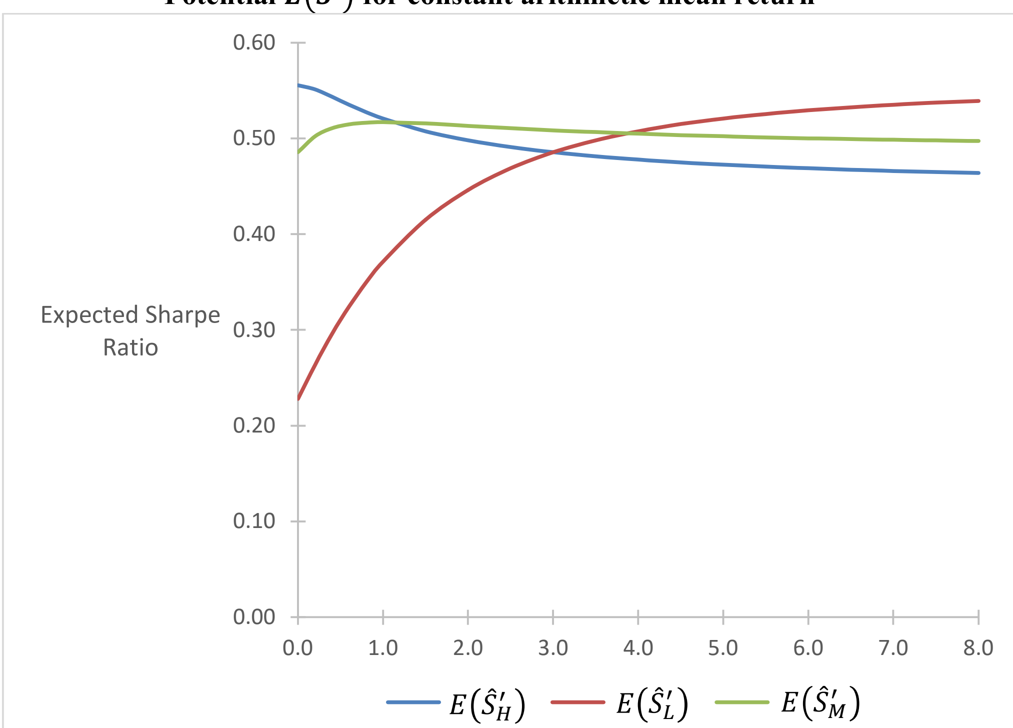
If the potential expected alphas can vary without altering returns, then the potential expected betas must vary in concert with the alphas in order to keep the expected returns fixed.

Figure 0.2
Potential $E(\widehat{\beta}')$ for constant arithmetic mean return



Similarly, for fixed average arithmetic returns, the expected Sharpe ratios can vary considerably.

Figure 0.3
Potential $E(\widehat{S}')$ for constant arithmetic mean return



The justification for these graphs comes easily and naturally from the new model, once it is understood.

Certainly, there are some extraordinary claims ahead.

Alpha Found Video 1

Introduction

Welcome to Alpha Found.

The goal of the following videos is to introduce and defend a new model for stock market dynamics. The primary way of demonstrating the correctness of the model will be to explain why low volatility portfolios tend to generate positive alphas.

It is the intent of these videos to transform the high alphas of low beta portfolios from a statistical anomaly into a mathematical expectation. It will take a while to get there and we will move slowly.

If you are watching this then you likely know that "alpha" has been the object of affection for active investors for decades. My favorite book on the subject is *Finding Alpha: The Search for Alpha When Risk and Return Break Down,* by Eric Falkenstein (2009, J. Wiley & Sons).

But there are many other books to consider such as *Searching for Alpha: The Quest for Exceptional Investment Performance* (by Ben Warwick, 2000, J. Wiley & Sons). And then there is *The Quest for Alpha* (by Larry E. Swedroe, 2011, Bloomberg Press). Notice the subtitle, "*The Holy Grail of Investing.*"

There are many more titles to consider such as:

> *Active Alpha: A Portfolio Approach to Selecting and Managing Alternative Investments*
> Alan H. Dorsey, 2007, Wiley & Sons, Inc.
>
> *The Alpha Formula: High Powered Strategies to Beat the Market with Less Risk*
> Larry Connors, Chris Cain, CMT, and Connors Research, LLC, 2020, The Connors Group
>
> *The Alpha Hunter: Profiting from Option LEAPS*
> Jason Schwarz, 2010, McGraw Hill Education
>
> *The Alpha Masters: Unlocking the Genius of the World's Top Hedge Funds*
> Maneet Ahuja, Myron Scholes, Mohamed El-Erian, 2012, J. Wiley & Sons
>
> *The Alpha Strategy: The Ultimate Plan of Financial Self-Defense*
> John A. Pugsley, 1981, Stanford Press
>
> *Alpha Trader: The Mindset, Methodology and Mathematics of Professional Trading*
> Brent Donnelly, 2021
>
> *Alpha Trading: Profitable Strategies That Remove Directional Risk*
> Perry Kaufman, 2011, J. Wiley & Sons
>
> *Better than Alpha: Three Steps to Capturing Excess Returns in a Changing World*
> Christopher M. Schelling, 2021, McGraw Hill Education
>
> *Delivering Alpha: Lessons from 30 Years of Outperforming Investment Benchmarks*
> Hilda Ochoa-Brillembourg, 2008, McGraw Hill Education

Finding Alphas: A Quantitative Approach to Building Trading Strategies
Igor Tulchinsky, 2019, J. Wiley & Sons

Geopolitical Alpha: An Investment Framework for Predicting the Future
Marko Papic, 2020, J. Wiley & Sons

Hedge Fund Alpha: A Framework for Generating and Understanding Investment Performance
John M. Longo, 2009, World Scientific Publishing Company

The Incredible Shrinking Alpha, 2nd Edition: How to be a Successful Investor Without Picking Winners, Larry E. Swedroe and Andrew L. Berkin, 2015, 2020, Harriman House

Organizational Alpha: How to Add Value in Institutional Asset Management
Ben Carlson, 2017, CreateSpace

Quantitative Strategies for Achieving Alpha
Richard Tortoriello, 2008, McGraw-Hill Finance & Investing

What's It All About Alpha? & Other Investment Essays from an Incredible Decade
Jason DeSena Trennert, 2016, Willow Street Press

Likely, there are some titles that were missed.

There are investor websites such as Seeking Alpha, and countless databases, consultants, and investment analytics companies measuring, reporting, and contemplating alpha.

All of this intellectual and computational output is for one reason: Alpha is excess return after adjusting for systematic risk. That's what everyone wants, excess risk-adjusted return. Positive alpha is the reward for having more luck, being smarter, or being faster than the other investors.

Historically, alpha has been strongly associated with low-risk stocks, but according to theory, it shouldn't happen that way. Any stock or portfolio of stocks should be just as likely as any other to have positive alpha, but that's not how it happens.

A great deal of work has gone in to understanding why it doesn't happen that way, but it remains a mystery, or at least many people find that the proposed explanations, however insightful, are ultimately unsatisfying. More on that in later videos.

The mystery of why low-risk stocks tend to generate high alphas has a name. It's called the "volatility effect" or the "low volatility anomaly." I'm going to call it the volatility effect because it is easier.

In addition to explaining the volatility effect, the proposed new model brings several other concepts into a new analytical framework. Concepts such as excess volatility; Blume betas; and zero-beta portfolios, which is known more to finance academics than to investment professionals, will be seen in a new and hopefully brighter light.

Most of those looking for alpha do so by looking for better insights into financial analysis, accounting data, or economic forecasts.

The new model is a mathematical and statistical model that was developed without reference to things such as ratio analysis; growth rates; free cash flow; the dividend discount model, or really any valuation model; fiscal and monetary policy; corporate governance; or technical analysis.

Not that there is anything wrong with those things, in fact there may be a lot right with them, they're just not a part of this model.

Before we get to the new model, there are a few things we'll need to cover: the background and literature, a definition of terms, including a careful definition of the volatility effect, alternate models, and finally, the assumptions of the capital asset pricing model and the consequences of relaxing those assumptions.

After that, once the new model is introduced, with a few exceptions, the vocabulary and mathematics will be in terms of elementary statistics.

In order to understand everything, you will need a fluency in elementary algebra; a fluency in elementary statistics including ordinary least squares regression; and a working knowledge of capital market theory.

For those possessing the prerequisites in abundance, it will be the 9th video before anything new or interesting appears on the screen.

The videos start out at an introductory and review level, although some more difficult ideas are referenced. In later videos, there will be plenty to chew on.

In the first several videos there are extensive references to the academic and popular literature. This is to give credit where credit is due, but also for those who want to chase down the sources, it will be easier knowing what the sources are.

See the website associated with these videos for a complete bibliography.

Finally, if you find an error, I would be grateful if you'd let me know at the email shown on the screen or found in the information below the video.

Alpha Found Video 2

Preliminaries

Welcome to part 2.

Before there was a theory of investing, there were just companies. People invested in companies based on the merits of those companies.

On January 6, 1900, Louis Bachelier, a mathematics Ph.D. student, had his thesis, *The Theory of Speculation* (Annales scientifiques de l'Ecole Normale Superiure, Ser 3,17 (1900), pg. 21-86) approved for publication. He defended his thesis on March 29, 1900, which is now considered the birthdate of mathematical finance. No one paid much attention at the time, but his thesis contained many ground breaking ideas concerning the way asset prices change.

Here is the first page of his thesis. If you don't want the French, here's the first page of an English translation.

Among the ground breaking ideas was the idea that, "The mathematical expectation of the speculator is zero." This isn't true in a strict sense, but it captures accurately the idea that since there is a buyer and a seller in each transaction, they can't both be right. Since both sides of a transaction are presumed to be intelligent, and there are many intelligent participants in the market, they should balance each other out over many transactions.

Bachelier thought that the outcomes of investment choices are random. Bachelier's thesis advisor, the great mathematician and physicist Jules Henri Poincare responded with, "What is chance to the ignorant is not chance to the scientist. Chance is only the measure of our ignorance."

Bachelier's largely forgotten thesis was rediscovered almost by chance in 1955 by future Nobel Prize winner Paul Samuelson. Samuelson was encouraged by Jimmie Savage to look for Bachelier's thesis. Savage, a mathematician, thought that economists would benefit from reading Bachelier.

Samuelson later, in 1965, brought some of the concepts into the modern context with a now famous paper called *Proof that Properly Anticipated Prices Fluctuate Randomly* (Industrial Management Review, 6 (2), 1965, 41-49).

Samuelson said of his own paper, "This theorem is so general that I must confess to having oscillated between regarding it as trivially obvious (and almost trivially vacuous) and regarding it as remarkably sweeping. Such perhaps is the characteristic of basic results." (*The Collected Scientific Papers of Paul A. Samuelson*, Volume III, 1972, Robert C. Merton, Ed. MIT Press).

Many investors, both then and now, believe there are cycles in stock market prices, seemingly contradicting the notion of prices fluctuating randomly. This might or might not be true, but Russian statistician Eugen Slutsky showed in 1927 that random processes can generate cyclical patterns that are indistinguishable from the cycles found in economic data.

Slutsky did not show that stock market price changes are random. He showed that cycles in some economic data are indistinguishable from cycles generated from the sum of random events. This is a subtle but important distinction.

Not many English-speaking economists were reading Russian language journals at the time. However, in

1937, Slutsky's work was translated and expanded in an article titled, *The Summation of Random Causes as the Source of Cyclic Processes*. Econometrica No. 5, April 1937, pg. 105.

In reference to the sum of random events, Slutsky said, (and this is a mouthful), "It is almost always possible to detect, even in the multitude of individual peculiarities of the phenomena, marks of certain approximate uniformities and regularities. The eye of the observer instinctively discovers on waves of a certain order other smaller waves, so that the idea of harmonic analysis… presents itself to the mind almost spontaneously."

Harmonic analysis means cycles, and cycles mean Fourier series. Fourier series takes us far afield from where we want to go, so we won't go there.

Here is the first page of Slutsky's 1937 paper.

There were other papers at the time on the mathematics of stock price movements, but just after 1950, mathematical methods for analyzing stock price changes began to flourish, particularly after Bachelier's thesis was rediscovered. Several researchers began to publish using ideas similar to, or in opposition to, Bachelier, each one trying to get at the mathematical nature of stock price movements.

Some of the more important papers from the 50s and 60s are:

> *The Analysis of Economic Time Series Part 1: Prices*
> Maurice G. Kendall, Journal of the Royal Statistical Society, Volume 116, 1953,
> Pgs. 11-25

This first one was originally titled *The Analytics of Economic Time Series*, but somehow the titled changed.

Next is:

> *Stock Market "Patterns" and Financial Analysis: Methodological Suggestions*
> Harry Roberts, Journal of Finance, Volume 14, 1959, Pgs. 1-10

Then we have:

> *Brownian Motion in the Stock Market*
> Matthew F.M. Osborne, Operations Research, Vol. 7, March-April 1959, Pgs. 145-173

Followed by:

> *A Revision of Previous Conclusions Regarding Stock Price Behavior*
> Alfred Cowles, Econometrica, Volume 28, October 1960, Pgs. 909-915

And:

> *Note on the Correlation of First Differences of Averages in a Random Chain*
> Holbrook Working, Econometrica, Volume 28, October 1960, Pgs. 916-918

Next is:

> *Some Characteristics of Changes in Common Stock Prices*
> Arnold B. Moore, this is a 1960 abstract of a 1962 dissertation, University of Chicago, as found in The Random Character of Stock Market Prices, 1964, Paul H. Cootner, ed.

Then:

Spectral Analysis of New York Stock Exchange Prices
Clive W.J. Granger and Oskar Morgenstern, Kyklos, Volume 16, February 1963, Pgs. 1-27

And:

Price Movements in Speculative Markets: Trends or Random Walks
Sydney Alexander, Industrial Management Review, Volume 2, May 1961, Pgs. 7-26

And a follow-up paper:

Price Movements in Speculative Markets: Trends or Random Walks Number 2
Sydney Alexander, Industrial Management Review, Volume 5, No. 2, 1964, Pgs. 25-46

Then we have:

Stock Prices: Random vs. Systematic Changes
Paul Cootner, Industrial Management Review, Volume 3, Spring 1962, Pgs. 24-45

Next is:

A Test of Non-randomness in Stock Price Changes
William Steiger, The Random Character of Stock Market Prices, 1964, MIT Press, Pgs. 253-261

And finally:

Periodic Structure in the Brownian Motion of Stock Prices
Matthew F.M. Osborne, Operations Research, Volume 10, Number 3, 1962, Pgs. 345-379.

You'll be happy to know; all these titles and several others can be found in a single volume:
The Random Character of Stock Market Prices, Paul Cootner, editor, published in 1964.

Unfortunately, it was out of print when I bought a used copy for about $70. It was reprinted in 2000, but as of the time of this writing the reprint is only available used for $100. First editions are quite a bit more.

There is a lot to learn from these articles. If nothing else you can learn just how sophisticated attempts were to get at the heart of what happens in financial markets. It's not as easy as it looks. But one thing you cannot learn from these articles is how to invest, they merely discuss evidence of what kind of mathematical structure best describes stock price movements. And they don't all agree.

It turns out that understanding stock price changes is slippery. Anyone who would suggest otherwise has either never worked with the data, or is selling something, or perhaps is fooling themselves.

This list of articles stops in 1964 for reasons that will be clear by the end of the video on the capital asset pricing model, two videos from now. However, very much has been published since then on the subject of the mathematics of stock price changes.

In two of the articles listed, the term "random walk" is used in the title. Random walks show up often in the mathematical finance literature.

Try this definition, "In mathematics a random walk is a mathematical object known as a stochastic or random process that describes a path that consists of a succession of random steps on some mathematical space such as integers."

I hope that was clear.

Well, formal definitions can sometimes obscure things. As a practical matter, “random walk” means that percentage price changes are random, and that each price change is unaffected by the price changes that came before it.

Concretely this means there are no trends or trend reversals, it is just the accumulation of noise that looks like a trend. If it’s true.

A popular non-mathematical book that discusses the subject, suitable for individual investors, is *A Random Walk Down Wall Street*, by Burton Malkiel (W.W. Norton & Company). The most recent edition was published in 2023, the fiftieth anniversary edition. Here is the 2007 edition from my library.

To what degree stock prices follow a random walk is still debated. It seems that the weight of the evidence is against the idea, but that weight might someday shift.

In any case, distinct from the book *A Random Walk Down Wall Street* is a book not so suitable for most individual investors called *A Non-Random Walk Down Wall Street* (by Andrew W. Lo & A. Craig MacKinlay, 2002, Princeton University Press). This book requires well-developed skill in statistics and time series to get much past the first page.

Understanding mathematics is one thing, investing is another. One of the two foundational works on how to select investments is, *The Theory of Investment Value* by John Burr Williams (Harvard University Press, 1938, republished in 1997 by Fraser Publishing Company).

Williams was the first to make a comprehensive case for valuing a stock based on the present value of future dividends, or more generally, the present value of future cash flows. His book relies on mathematical reasoning throughout.

The other foundational work is more popular among value investors: *Security Analysis* by Benjamin Graham and David Dodd. Fortunately for many investors, *Security Analysis* is not as mathematically oriented as *The Theory of Investment Value*.

Here is the first edition published in 1934, and here is the second edition published in 1940, and here is the 3rd edition published 1951. Each of the first three editions was reprinted in 1997, 2002, and 2004, respectively.

Aside from the three editions which have been reprinted, there is the 4th edition, published in 1962 (sorry, no picture), the 5th edition published in 1988, (again, sorry no picture), and for your viewing pleasure here is the 6th edition published in 2009.

First editions can sell for several thousand dollars. First printings of the first edition commonly list for over $20,000, although I don’t know if they actually sell for that much. Obviously, there is a lot of loyalty to the book and to the principles it promotes.

People who invest using the methods described in *Security Analysis* are often called “Graham-Dodd bottom-up investors.”

Still, although there were ideas about what stocks to invest in individually (those that met the criteria of these two books), and ideas about the mathematical structure of price movements, there was no theory of how to build a portfolio, other than general advice to diversify.

Diversification was a seat of the pants type procedure. There was no formal method you could use to know when you have arrived.

William Shakespeare is often cited as one inspiration for diversification, from the Merchant of Venice, Act 1, Scene 1.

> My ventures are not in one bottom trusted,
> Nor to one place,
> Nor is my whole estate upon the fortunes of this present year.
> Therefore, my merchandise makes me not sad.

The word "bottom" in the first line refers to the hold of a ship where cargo was stored, but maybe that was obvious.

Those who take no inspiration from Shakespeare, may find it in the Bible.

> Ecclesiastes 11:2 Divide your portion to seven, or even to eight, for you do not
> know what misfortune may occur on the earth. (New American Standard Bible, 1995)

Enter Harry Markowitz.

Alpha Found Video 3

Harry Markowitz

Welcome to Alpha Found part 3.

Despite appearances these videos are not about the history of investing, but in order to understand the current state of capital market theory, some history is required as background. A large part of that history is about the contributions of Harry Markowitz.

The question of how to accomplish diversification was not yet solved in the early 1950s. As inspiration for solving the problem of diversification, Markowitz famously observed, "I was struck by the notion that you should be interested in risk as well as return."

In 1952, Harry Markowitz, a 25-year-old economics Ph.D. student at the University of Chicago published a 14-page paper in the Journal of Finance with the modest title, *Portfolio Selection (Journal of Finance, March 1952, Pgs. 77-91)*.

Here is the first page.

In 1959 Markowitz published a book by the same title as a gentle introduction to his theory. A full and not so gentle explanation of his theory was published in 1987 under the title *Mean-Variance Analysis in Portfolio Choice and Capital Markets*. The problem of diversification had been solved.

Eventually that short paper of 1952 would earn Markowitz the Nobel Prize in Economics.

Return is easy to understand; it's the percent change in the value of your investments over time. Risk is tougher to define, but Markowitz decided that variance of return is the proper measure of risk.

Most investors don't think "variance of return" when they think of risk, but nonetheless, it's a good definition in that it is mathematically precise and often easy to work with.

There are several assumptions underlying Markowitz:

1. Investors are concerned with only two parameters: risk and return.
2. Investors are risk averse.
3. Investors seek the highest level of return for a given level of risk.
4. All investors have the same expectations concerning return, variance, and covariance for all risky assets. This is usually called *homogeneous expectations*.
5. All investors have the same 1 period time horizon.
6. Stock returns follow a joint normal distribution.

Markowitz has since said that joint normal is not needed, any of over 100 distributions will work, but joint normal is often used since it is the easiest distribution to work with.

The assumptions for Markowitz can be found in many places. The first five assumptions in this case came from *Finance: Capital Markets, Financial Management, and Investment Management*, by Frank Fabozzi and Pamela Peterson Drake (J. Wiley & Sons, 2009, pg. 596).

Unfortunately, I have the book in PDF so I can't show you where it is on my bookshelf.

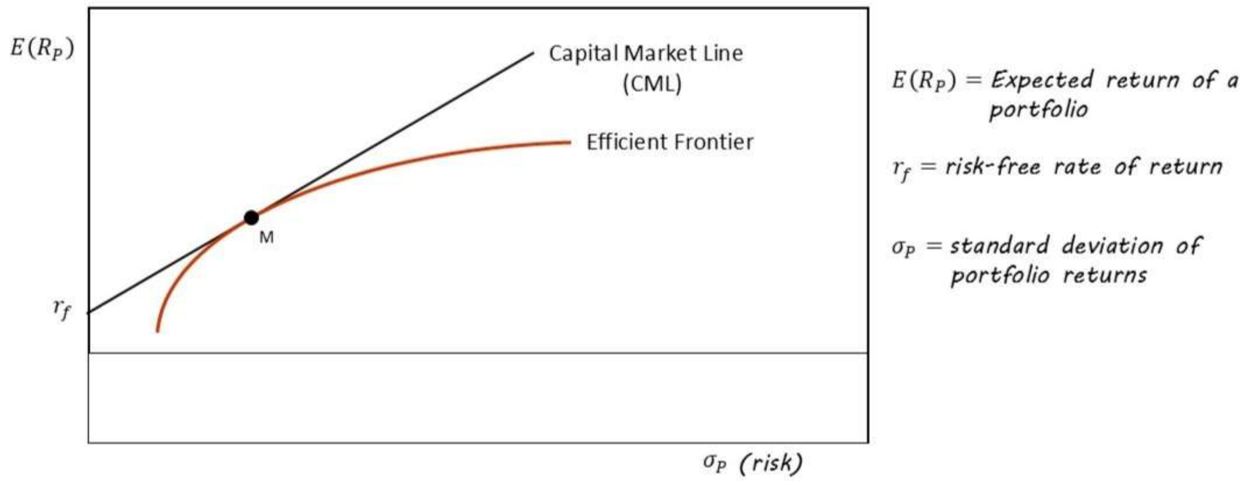
So, Markowitz solved the problem of diversification, and as such, the problem of how to create an investment portfolio, sort of. We will get to why he only sort of solved the problem in a couple of minutes.

His method is often referred to as mean-variance optimization or E-V optimization, where the E is the mathematical expectation operator. Expectation just means mean, and V is variance.

Mean-variance optimization as applied to investments is a specific application of quadratic optimization, which used to be called quadratic programing. When applied to investing, mean-variance optimization is called Modern Portfolio Theory or MPT.

By the way, I will use variance and standard deviation interchangeably when it comes to discussions of risk, but not interchangeably when we finally get to calculations in later videos.

Figure 3.1
The Efficient Frontier



But what does it mean? Essentially, the idea is that with all of the potential investments or combinations of investments that someone might choose, an investor would want to minimize risk for any expected return, or maximize expected return for any given level of risk. A formula will not accomplish the desired calculation. Instead, an algorithmic procedure is used.

There are an infinite number of possible portfolios someone might construct, but the best portfolio for any given level of expected return will have the lowest possible expected standard deviation of returns. In this case, what that means is that although you will not likely get the expected return exactly, but some level of return above or below that amount, the standard deviation of the possible returns will be at a minimum when compared with the standard deviation of other portfolios with the same expected return. Alternatively, the best portfolio is the one that gives the maximum expected return for the amount of risk. The result of Markowitz mean-variance analysis is what you see on the screen.

The red curve is called the efficient frontier. The M represents the market portfolio, which is assumed to be one of the efficient portfolios. Any point on the curve represents an efficient portfolio, and any point below the curve represents an inefficient portfolio, meaning either lower risk is available for the same expected return, or higher expected return is available for the amount of risk.

The vertical axis is the expected return of the portfolio. About a quarter of the way up is r_f, which is the risk-free rate of return (usually US government bills or bonds). The horizontal axis uses standard deviation for a risk measure. The line drawn from the risk-free rate through M is called the capital market line.

The implication is that it is best to own some combination of the market portfolio and the risk-free asset. To achieve a risk/return combination different from the market by buying a different efficient portfolio on the curve, will be less efficient than buying the market portfolio with a portion of your assets and placing the rest of your assets in the risk-free asset. The portion of the line extending beyond the market portfolio, up and to the right, represents borrowing, in order to put more than 100% of your assets in the market. Put another way, the implication is that whatever percentage you put in the stock market, that percentage should be put in the market portfolio, and the remaining percentage should be put in the risk-free asset.

The specifics of the algorithm that constructs the efficient frontier is not important for these videos. However, the inputs alone are challenging enough. If someone wanted to construct a mean-variance optimization for the stocks that make up the S&P 500, the following inputs are needed:

Estimated mean returns for each of 500 stocks,
Estimated variance for each stock, and estimated covariance between each pair of 500 stocks.

All told that is 500*501/2 = 125,250 covariances, plus 500 estimated mean returns, or 125,750 estimates of future price behavior in total.

If you wanted to use historical data to make the estimates, and 5 years of monthly data is used. You have 500 x 60 = 30,000 data points to make the 125,750 statistical estimates. Therefore, there is a massive problem of insufficient degrees of freedom. As a result, there is no good way to make the needed individual estimates from historical data.

Apart from the impossibility of making such estimates in the first place, even if it could be done, no computers available in 1952 could then handle the computations that would follow. The great insights of Harry Markowitz would have to wait for further insights to be practical.

As an aside, today these calculations are easily done at the level of asset classes or economic sectors. If the asset classes are few, there is no degrees of freedom problem when historical statistical relationships are used to estimate future statistical relationships, if the relationships are stable.

Much more could be said about the application of mean-variance efficiency to investment theory and practice, but we need to push forward to the Capital Asset Pricing Model and William F. Sharpe.

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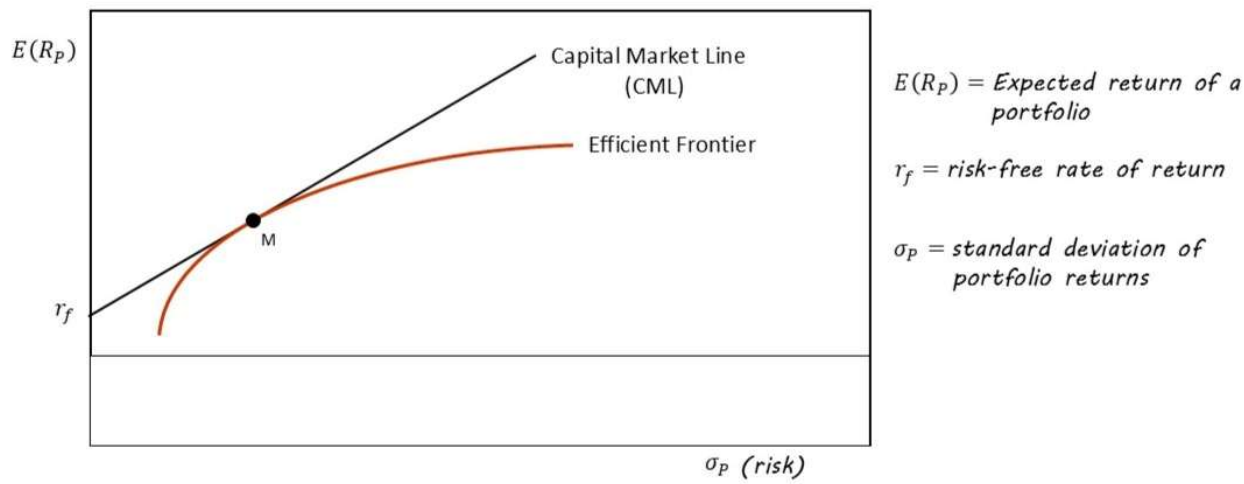
His method is often referred to as mean-variance optimization or E-V optimization, where the E is the mathematical expectation operator. Expectation just means mean, and V is variance.

Mean-variance optimization as applied to investments is a specific application of quadratic optimization, which used to be called quadratic programing. When applied to investing, mean-variance optimization is called Modern Portfolio Theory or MPT.

By the way, I will use variance and standard deviation interchangeably when it comes to discussions of risk, but not interchangeably when we finally get to calculations in later videos.

Figure 3.1
The Efficient Frontier



But what does it mean? Essentially, the idea is that with all of the potential investments or combinations of investments that someone might choose, an investor would want to minimize risk for any expected return, or maximize expected return for any given level of risk. A formula will not accomplish the desired calculation. Instead, an algorithmic procedure is used.

There are an infinite number of possible portfolios someone might construct, but the best portfolio for any given level of expected return will have the lowest possible expected standard deviation of returns. In this case, what that means is that although you will not likely get the expected return exactly, but some level of return above or below that amount, the standard deviation of the possible returns will be at a minimum when compared with the standard deviation of other portfolios with the same expected return. Alternatively, the best portfolio is the one that gives the maximum expected return for the amount of risk. The result of Markowitz mean-variance analysis is what you see on the screen.

The red curve is called the efficient frontier. The M represents the market portfolio, which is assumed to be one of the efficient portfolios. Any point on the curve represents an efficient portfolio, and any point below the curve represents an inefficient portfolio, meaning either lower risk is available for the same expected return, or higher expected return is available for the amount of risk.

The vertical axis is the expected return of the portfolio. About a quarter of the way up is r_f, which is the risk-free rate of return (usually US government bills or bonds). The horizontal axis uses standard deviation for a risk measure. The line drawn from the risk-free rate through M is called the capital market line.

The implication is that it is best to own some combination of the market portfolio and the risk-free asset. To achieve a risk/return combination different from the market by buying a different efficient portfolio on the curve, will be less efficient than buying the market portfolio with a portion of your assets and placing the rest of your assets in the risk-free asset. The portion of the line extending beyond the market portfolio, up and to the right, represents borrowing, in order to put more than 100% of your assets in the market. Put another way, the implication is that whatever percentage you put in the stock market, that percentage should be put in the market portfolio, and the remaining percentage should be put in the risk-free asset.

The specifics of the algorithm that constructs the efficient frontier is not important for these videos. However, the inputs alone are challenging enough. If someone wanted to construct a mean-variance optimization for the stocks that make up the S&P 500, the following inputs are needed:

> Estimated mean returns for each of 500 stocks,
> Estimated variance for each stock, and estimated covariance between each pair of 500 stocks.

All told that is 500*501/2 = 125,250 covariances, plus 500 estimated mean returns, or 125,750 estimates of future price behavior in total.

If you wanted to use historical data to make the estimates, and 5 years of monthly data is used. You have 500 x 60 = 30,000 data points to make the 125,750 statistical estimates. Therefore, there is a massive problem of insufficient degrees of freedom. As a result, there is no good way to make the needed individual estimates from historical data.

Apart from the impossibility of making such estimates in the first place, even if it could be done, no computers available in 1952 could then handle the computations that would follow. The great insights of Harry Markowitz would have to wait for further insights to be practical.

As an aside, today these calculations are easily done at the level of asset classes or economic sectors. If the asset classes are few, there is no degrees of freedom problem when historical statistical relationships are used to estimate future statistical relationships, if the relationships are stable.

Much more could be said about the application of mean-variance efficiency to investment theory and practice, but we need to push forward to the Capital Asset Pricing Model and William F. Sharpe.

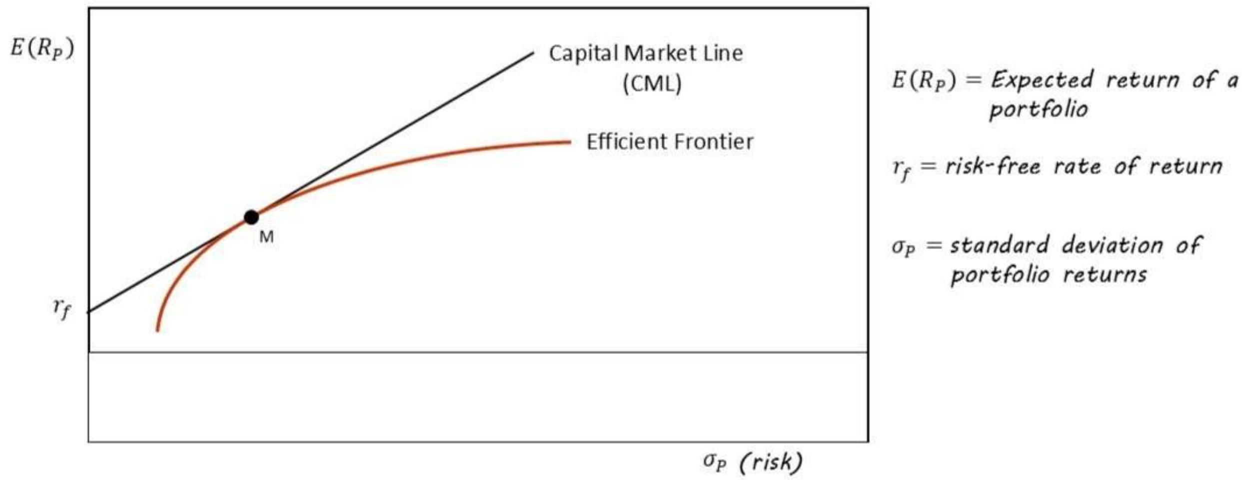
Alpha Found Video 4

CAPM

Welcome to part 4 of Alpha Found.

The insights initiated by Harry Markowitz in 1952 lacked a practical implementation. The curve known as the efficient frontier was either impossible or impractical to implement.

Figure 3.1
The Efficient Frontier



Markowitz was working for the RAND corporation in 1960 when William Sharpe started working there also. Sharpe was looking for a Ph.D. topic. At the suggestion of Markowitz, Sharpe took on the single factor model. That is, instead of estimating thousands of expected covariances, possibly there is one economic factor that each security has a sensitivity to. If that sensitivity can be estimated, then the elaborate mean-variance optimization algorithm becomes unnecessary.

Sharpe's solution to the single factor model was published under the title *Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk*, in the September, 1964 edition of The Journal of Finance, Pgs. 425-442.

And here is the first page.

The single factor that matters is the excess return on the market portfolio itself. That is, it is the collection of all risky assets, held in proportion to their market capitalization. The single factor model is called the Capital Asset Pricing Model or CAPM. It was the exact breakthrough that everyone needed.

I should add for the sake of historical accuracy that others are often credited with coming up with the same theory, or ideas very similar, and thus contributed to CAPM's development. Those would be Jack Treynor, who had the idea before Sharpe but did not publish, John Lintner, and Jan Mossin. CAPM is often referred to as the Sharpe-Lintner-Mossin Capital Asset Pricing Model.

How sensitive the return of a stock or portfolio is to the excess return of the market is called beta, which I am calling CAPM beta. The notation I am using is beta prime (β').

The prime notation is my own. I use it because in later videos there will be different parameters and variables that look and operate similarly to one another but are meaningfully different, sometimes within the same equation.

As a notational convention, anything with a prime mark has the ordinary meaning as found in the context of CAPM. Anything without the prime is defined when it is first used. Right now, it looks like an unnecessary complication.

If the total return of the market is R'_M, then CAPM beta is the sensitivity to the total return of the market minus the risk-free rate. So, that $R'_M - r_f$ is called the excess return because it is in excess of the risk-free rate of return.

The information needed to calculate the change in market value for all risky assets is unobtainable. Various efforts have been made to make an estimate of a market of all risky assets, but at the end of the day, usually a broad market index is used instead of a hypothetical market of everything. For example, the S&P 500 will be used for this purpose in later videos of this series.

Oddly enough, Sharpe's paper does not include the formula made famous by his paper. All that is given is a verbal description. When written out, for a single time period it looks like this:

$$R'_{at} = r_f + \hat{\beta}'_a\left(R'_{Mt} - r_f\right) + \hat{\varepsilon}'_{at} \tag{4.1}$$

Where:

R'_{at} = the return of an asset or portfolio in period *t*.

r_f = the risk-free rate of return.

$\hat{\beta}'_a$ = the estimated sensitivity of the asset to the excess market return.

R'_{Mt} = the market return including income in period *t*.

$R'_{Mt} - r_f$ = the excess return in period *t*.

$\hat{\varepsilon}'_{at}$ = idiosyncratic return in period *t*.

and finally, $\hat{\varepsilon}'_{at}$ is the return in period *t* not explained by the rest of this model, called idiosyncratic return or residuals in the context of regression.

And as an average over several time periods, it looks like this:

$$\bar{R}'_a = r_f + \hat{\beta}'_a\left(\bar{R}'_M - r_f\right) \tag{4.2}$$

Where $\bar{R}'_a$ is the average return of an asset over several time periods, and $\bar{R}'_M$ is the average market return.

If we move from average returns over several time periods (ex-post), to expected returns (ex-ante) we get equation (4.3).

$$E(R_{at}) = r_f + \beta'_a\left(R'_{Mt} - r_f\right) \tag{4.3}$$

Where *E* is the expectation operator from probability, which just means expected value or mean.

Usually, the risk-free rate is moved to the left-hand side of the equation to give equation (4.4).

$$E(R_{at}) - r_f = \beta_a' \left(R_{Mt}' - r_f \right) \tag{4.4}$$

So, the original CAPM is an equation of a line with zero intercept. Equations (4.1) and (4.2) are ex-post versions of CAPM, while (4.3) and (4.4) are ex-ante versions. Probably the most frequently used expression is equation (4.4).

When a line number is in blue, it means we have seen it before.

But where does estimated CAPM beta come from? How do we know its value?

In practice, CAPM beta is an unknown, but if we did know it, then our problems would be solved, or so many people believe. CAPM beta is considered to be a parameter, unchanging over some relevant time period. Since we do not know the value of CAPM beta, it has to be estimated.

The estimation technique for CAPM beta is ordinary least squares regression, which means that estimated CAPM beta, $\hat{\beta}_a'$, is the slope of a regression line. Usually, the ex-post estimate of the last several time periods becomes the ex-ante estimate of the next time period. Also, usually when people are simply talking, CAPM beta and estimated CAPM beta are used interchangeably, but in a formal context, they are different things.

Put another way, although I will not give a statistics lecture in the middle of this video, let me say that as a generality, we would like to know the true value of CAPM beta, but we don't. We do know the estimated value from the data, which is our best guess or estimate as to the true value. With that said, the expected value of $\hat{\beta}'$ is equal to β', or at least it is supposed to be. After a few more videos, knowledge of elementary statistics will be assumed; so knowledge of things like the difference between parameters and parameter estimates will also be assumed.

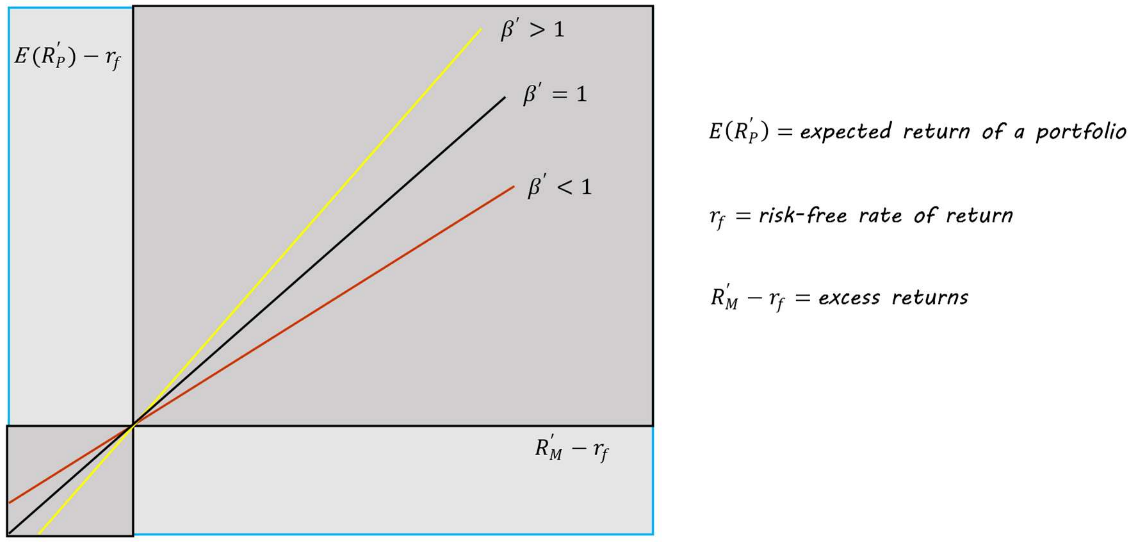
CAPM assumes a zero intercept, meaning the regression technique is the zero-intercept regression.

As a stylistic point, I have a hard time putting the expectation operator in front of the market return as textbooks do. "Expectation" has a specific mathematical meaning. Market returns do not have a mathematical expectation. With that said, once the market returns are known, the expectation operator does apply to the return of the asset.

The risk, or standard deviation, of the excess returns of the market is called systematic risk. In addition to scaling market returns, β' also scales systematic risk. Much more on that as the videos progress.

Here is a graphical representation of CAPM.

Figure 4.1
The Capital Asset Pricing Model



If $\beta' > 1$, then the asset responds to market changes more strongly than the market itself. If $\beta' < 1$, then the asset responds to market changes less strongly than the market itself. The excess return of the market has a CAPM beta of one by definition. Anything regressed against itself will have a slope of one.

The assumptions needed to make CAPM work are as follows (this is from Investments by Bodie Kane and Marcus, 10th edition):

The first three have to do with investor behavior.

1a) Investors are rational, mean variance optimizers.
1b) Investors' planning horizon is a single time period.
1c) Investors have homogeneous expectations.

The next four have to do with market attributes.

2a) All assets are publicly held and trade on public exchanges. Short positions are allowed, and investors can borrow or lend at the risk-free rate.

This seems like three assumptions to me, not one, but this is how Bodie, Kane, and Marcus do it.

2b) All information is publicly available (this is called informational symmetry).
2c) No taxes.
2d) No transaction costs.

Here is a picture of the tenth edition if you need one.

It is easy to find a different list of assumptions for CAPM in different sources, but there is always substantial overlap.

These assumptions are not great, but they got things moving forward. Once they got moving forward, CAPM took on a life of its own. We will return to these assumptions in a later video.

In order to make an estimate of CAPM beta, returns of a benchmark and of the asset of interest are needed for several time periods. For statistical validity, usually at least 30 time periods are used. Often, monthly data for 5 years is used giving 60 data points. This is enough data to smooth out the bumps along the way. Some use 3 years or 36 months because CAPM beta may not be stable over longer periods of time.

There is nothing magic about using months. You could use quarters, or years if the parameters were stable, or weeks, but most use months.

Returns include returns from dividends.

Shortly after CAPM was published, Michael Jensen, another Ph.D. economics student at the University of Chicago, and professor at the University of Rochester, suggested an improvement.

Alpha Found Video 5

Jensen's Alpha

Welcome back to part 5 of Alpha Found.

At the 26th annual meeting of the American Finance Association, held December 28-30, 1967, Michael Jensen introduced the idea of including the intercept in the regression equation as a method of evaluating investment performance. The intercept is called alpha, and so alpha was born. This is where all the current excitement about alpha stems from.

In Jensen's paper from the meeting proceedings, published in the Journal of Finance, May 1968, use of the intercept was spelled out for those who did not attend the meeting. Notice the very uninteresting title: *The Performance of Mutual Funds in the Period 1945 – 1964* (Journal of Finance, Volume 23 No. 2, Pgs. 389-416).

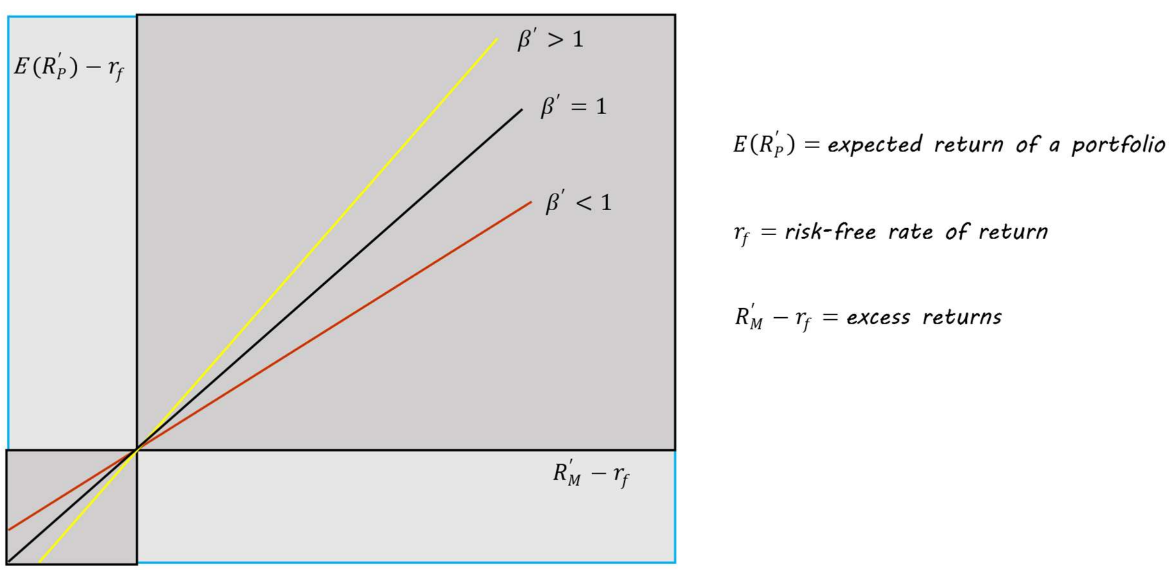
Here is the first page.

The big breakthrough is highlighted here in equation 8. The intercept of a regression line is called alpha. Alpha can be positive or negative, but the market has a zero alpha. Investors want a positive alpha, not just alpha generically. Alpha is interpreted as excess return after adjusting for systematic risk.

From the point of view of more than 50 years later, it seems obvious. Sharpe and Jensen made it obvious. It was not obvious at the time. And ever since, for more than 50 years, investors have been seeking positive alpha. As a result of all the talk of alpha, now when people discuss CAPM it includes alpha, though it wasn't there initially.

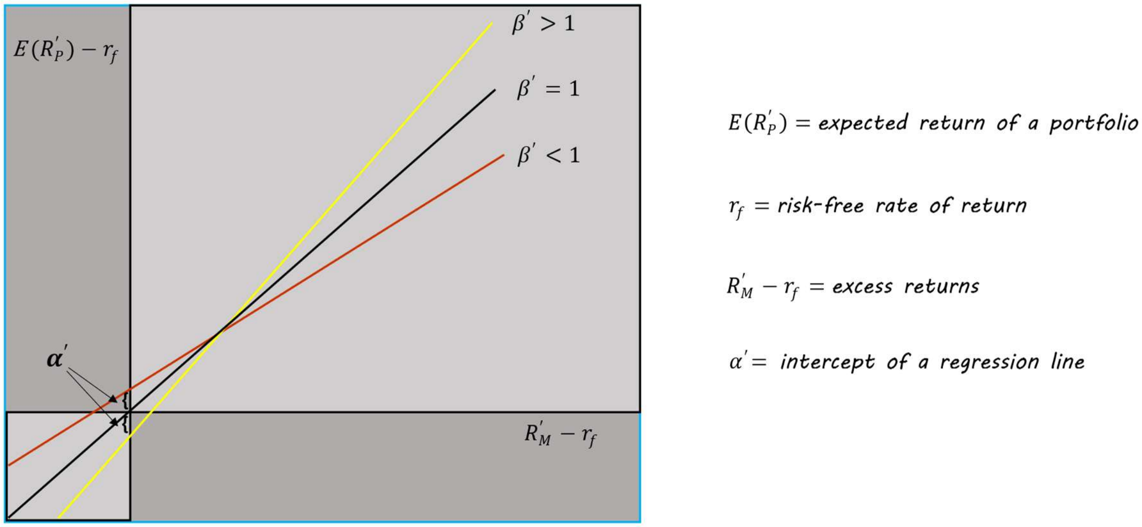
Here is a graph of Sharpe's original idea of CAPM as seen in the last video.

Figure 4.1
The Capital Asset Pricing Model



Here's what it looks like with alpha, using notation α'. Alpha is the intercept.

Figure 5.1
CAPM with Alpha



Now the different ways of expressing the equations of the capital asset pricing model look like this:

$$R'_{at} - r_f = \hat{\alpha}'_a + \hat{\beta}'_a(R'_{Mt} - r_f) + \hat{\varepsilon}'_{at} \tag{5.1}$$

$$E(R'_{at}) - r_f = E(\hat{\alpha}'_a) + \beta'_a(R'_{Mt} - r_f) \tag{5.2}$$

$$E(\bar{R}'_a) - r_f = E(\hat{\alpha}'_a) + \beta'_a(\bar{R}'_M - r_f) \tag{5.3}$$

With the usual components plus the addition of a term called $\hat{\alpha}'_a$, which is the estimated excess return of the asset after adjusting for systematic risk.

R'_{at} = the return of an asset in period t.

r_f = the risk-free rate of return.

$\hat{\alpha}'_a$ = estimated excess return of an asset after adjusting for systematic risk, called "alpha."

$\hat{\beta}'_a$ = an asset's estimated sensitivity to excess returns.

R'_{Mt} = total return of the market index in period t.

$R'_{Mt} - r_f$ = total excess return of the market index in period t.

$\hat{\varepsilon}'_{at}$ = the estimated return not explained by the model, called idiosyncratic return.

In addition to expected returns, we can look at risk. To find the risk of a portfolio, find the variance of the ex-post CAPM equation, including the intercept. So, we need to take the variance of both sides of equation (5.1).

$$var(R'_a - r_f) = var(\hat{\alpha}'_a + \hat{\beta}'_a(R'_M - r_f) + \hat{\varepsilon}'_a)$$

Isolated constants drop out of variance calculations, and constant coefficients get pulled out and squared.

$$var(R'_a) = var(\hat{\beta}'_a(R'_M) + \hat{\varepsilon}'_a) = \hat{\beta}'^2_a var(R'_M) + var(\hat{\varepsilon}'_a)$$

The right-most term is called idiosyncratic risk. Much more on that to come. This assumes that idiosyncratic risk is independent from systematic risk.

And finally, with a change in notation we have equation (5.4).

$$\hat{\sigma}_a'^2 = \hat{\beta}_a'^2 \hat{\sigma}_M'^2 + \hat{\sigma}_{a\varepsilon}'^2 \tag{5.4}$$

Equation (5.4) is the equation for the ex-post variance of a portfolio using CAPM regression components.

You'll have to excuse the change in notation. Some notation is easier to work with and other notation is easier to look at.

Diversification with CAPM means trying to minimize idiosyncratic risk for a given portfolio β_P'.

Alpha measures the return that does not come from systematic risk. If it doesn't come from systematic risk, where does it come from? The big idea is that alpha, positive alpha, reflects skill in selecting investments. It's a measure of value added.

The theory behind alpha is that the expected alpha of any investment is zero.

$$E(\hat{\alpha}_a') = 0 \tag{5.5}$$

Randomness in investment returns means that some stocks will have positive alpha, some negative, but in either case it's a result of randomness in the outcomes, not skill. The weighted sum of all alphas is necessarily 0 in CAPM because the market regressed against itself will have a zero intercept, which is zero alpha.

There is a hypothesis called the joint hypothesis, framed by Eugene Fama in 1970, which states that capital markets are efficient, and the capital asset pricing model is the correct model for measuring efficiency. If the joint hypothesis is true, then alpha is random and centered on zero.

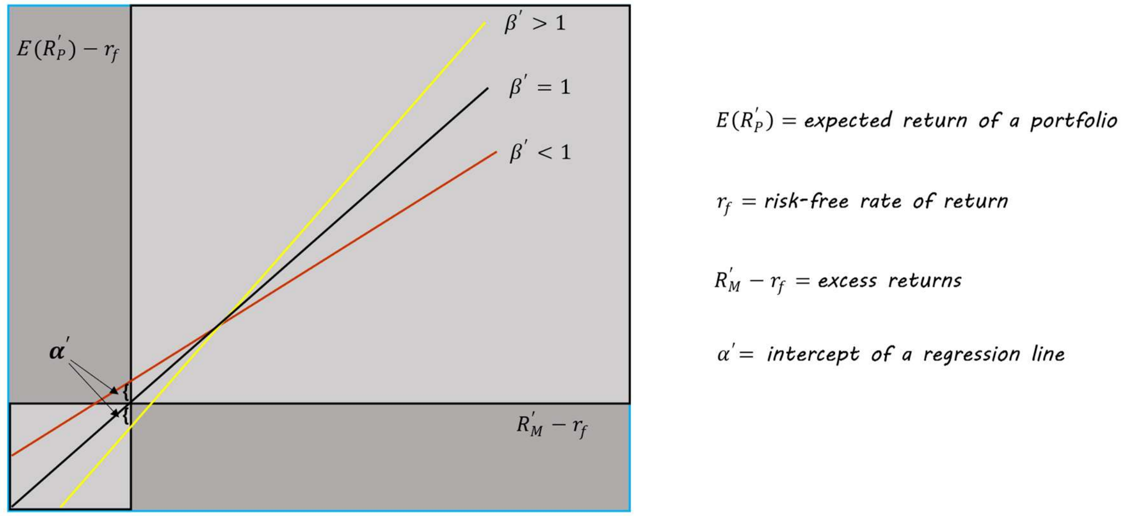
The pressing question concerns the randomness of alpha. If an investor (or algorithm for that matter) can generate positive alpha such that alpha is statistically significant, then that statistical significance stands as evidence of investor skill. Since under CAPM, the total alpha is zero by construction, the real question is not whether it is centered on zero, but do the component alphas deviate randomly from zero?

Although a few people quickly wrote papers testing alpha, the first comprehensive paper was by Fisher Black, Michael Jensen, and Myron Scholes, a paper now referred to as Black, Jensen, and Scholes, or BJS. The formal title of BJS is *The Capital Asset Pricing Model: Some Empirical Tests*, published in a 1972 book titled *Studies in the Theory of Capital Markets* (MC Jensen, Ed.).

BJS found unambiguously that positive alpha is associated with low CAPM beta, and negative alpha, therefore, is associated with high CAPM beta. This isn't lock-step certainty, but a statistical tendency. Importantly, they eliminated ex-post selection bias as an explanation. That is, it is not only true when looking back in time after we have the answers, it's also predictive.

Another look at this graph shows the effect, where $\beta' < 1$ implies that $E(\hat{\alpha}') > 0$; and $\beta' > 1$ implies $E(\hat{\alpha}') < 0$.

Figure 5.1
CAPM with Alpha



Certain aspects of BJS will be looked at closely in a later video, but for now the important part is the conclusion that positive alpha and low CAPM beta are closely associated. The relationship shows up in every decade since the 1940s, long before there was a model to measure it. According to BJS, the 1930s did not have the relationship, but the time period covered was during the great depression when stock price movements were unusual and large.

How is this possible?

After BJS the next big paper is usually called Fama-McBeth published in 1973, meaning Eugene Fama and James McBeth. The actual title is *Risk, Return, and Equilibrium: Empirical Tests* (Journal of Political Economy, Volume 81, May-June 1973, Pgs. 607-636). And then there was Haugen and Heins published in 1975, meaning Robert A. Haugen and A. James Heins. They wrote *Risk and the Rate of Return on Financial Assets: Some Old Wine in New Bottles* (Journal of Financial and Quantitative Analysis, Volume 10, December 1975, Pgs. 775-784).

There were many other lesser known but important papers written at this time. In addition to the authors already mentioned, George Douglas, Marshall Blume, and Irwin Friend are significant, as well as others.

Markowitz, Sharpe, Scholes, and Fama, among others we will run across later, received Nobel Prizes in economics for their work on capital market theory. Samuelson also received a Nobel Prize in economics, but not for his contributions to capital market theory, although his contributions were significant. Fisher Black would have also been included on the list, but he passed away before his collaborator, Myron Scholes, was awarded the prize.

Every once in a while, I run into someone who thinks it is easy to get outsized investment returns. The fact that so many people were awarded Nobel Prizes for their work on capital markets stands as testimony that the issues involved are complex and deep.

The idea that low CAPM beta is associated with positive alpha is so thoroughly documented in the academic literature that I am going to assume that the relationship exists and that it is durable over time.

There is no use trying to prove what is not in doubt, but this fact really makes very little sense at the level of economic theory. Economic theory would tell us that what generates returns is systematic risk, yet the positive alpha - low CAPM beta association stands in contrast to this.

So we have found alpha in low CAPM beta stocks. The big question is why is it found there? There aren't that many possibilities available. Either CAPM or its assumptions is flawed (some might say misspecified), or the model is right but the parameter estimation method for alpha and CAPM beta is flawed, or markets are not efficient, or some combination of these three.

Alpha Found Video 6

Alternative Models – Multi-Moment Models

Welcome back to Alpha Found.

The last video ended with three possibilities that could explain why positive alpha is associated with low CAPM beta. They are: CAPM is flawed, or the estimation methods for alpha and CAPM beta are flawed, or markets are not efficient, or some combination of these three.

To look at this further let's look first at alternative models.

There are two types of models that serve as alternatives to CAPM but in some way are derivative of CAPM. There are multi-moment models, and multi-factor models.

MPT uses expected mean returns and the covariance between securities. CAPM uses mean returns and the covariance with the market. Mean return is called the first moment, variance is the second moment. Purists would call variance the second central moment.

The third and fourth central moments do not show up in MPT and CAPM.

The third central moment is skewness, which measures the degree to which data is distributed asymmetrically, including outsized returns to the upside or downside. The hope for outsized returns, or lack thereof, might alter what investors are willing to pay for an asset. It is doubtful that most investors know what skewness is in a formal sense, and much more doubtful that they have a working knowledge of how it operates in the context of portfolio construction. From before the beginning of formal asset pricing theory, skewness was observed in price dynamics.

The fourth central moment is kurtosis which means "fat tails" or "long tails" of a statistical distribution.

Is it possible that investors are consciously or unconsciously considering higher moments when the make investment choices, including choices concerning diversification?

The book *The Random Character of Stock Market Prices*, discusses skewness. Yet, when formal theory was developed, higher moments were ignored. Easier problems had to be solved before more difficult problems could be tackled. Higher moment optimization was thought to be intractable.

No one believes that stock returns are normally distributed, but the normal distribution is so easy to work with that it is assumed good enough for a lot of modeling purposes. Only two moments are needed to describe completely the normal distribution, mean and variance.

If a greater sophistication than the normal distribution is desired, the next distribution of choice after the normal is the lognormal. It has a better intuition to it, and it has well defined skewness. However, it is much harder to work with in many applications. Also, like the normal, the lognormal ultimately doesn't work as well as we would like.

If the level of sophistication is upped again, some, like Benoit Mandelbrot, favor the stable family of distributions, which has certain advantages, but also has some absurdities when applied to investing, like infinite variance. It should be noted that the normal distribution is the only distribution in the stable family that has a finite variance.

It is a conundrum as to what probability distribution applies. I will have a thought about this in the 28th video.

We haven't discussed the formula for CAPM beta yet so let me do that now.

In the CAPM OLS case:

$$\hat{\beta}'_a = \frac{\hat{\sigma}'_{a,M}}{\hat{\sigma}'^2_M} = \hat{\rho}'_{a,M}\frac{\hat{\sigma}'_a}{\hat{\sigma}'_M} \tag{6.1}$$

But this is a two-moment model.

Even if the mathematics of multi-moment pricing models had been worked out in the early days, the estimates needed to run the optimization are significant.

Just as MPT could be simplified to a single factor model called CAPM, there is a single factor simplification of the multi-moment model. It introduces an additional parameter called constant proportional risk aversion.

In some multi-moment models:

$$\hat{\beta}'_a = \frac{\hat{\sigma}'_{a,-M^{-b}}}{\hat{\sigma}'_{M,-M^{-b}}} \tag{6.2}$$

Where:

b = constant proportional risk aversion

$\hat{\beta}'_a$ = multi-moment beta of the asset

$\hat{\sigma}'_{a,M}$ = covariance between the asset and the market

$\hat{\rho}'_{a,M}$ = correlation between the asset and the market

$\hat{\sigma}'_{a,-M^{-b}}$ = covariance between the asset and the negative of the market to the negative b power

$\hat{\sigma}'_{M,-M^{-b}}$ = covariance between the market and the negative of the market to the negative b power

The formula does not have much intuition to it.

Under certain conditions constant proportional risk aversion (b) can equal 1. With that simplifying assumption, we get equation (6.3):

$$\hat{\beta}'_a = \frac{\hat{\sigma}'_{a,-M^{-1}}}{\hat{\sigma}'_{M,-M^{-1}}} \tag{6.3}$$

One view of multi-moment models is that investors are pricing more moments than Modern Portfolio Theory or CAPM. The differences between the two moments of MPT and multi-moments models result in a positive alpha for low CAPM beta stocks because MPT is not considering all information important to investors such as skew and kurtosis. Once the higher moments are considered, then alpha disappears, or is supposed to disappear.

A different interpretation is that the pursuit of higher moments by investors, who can be risk seeking rather than risk averse, causes the prices of higher CAPM beta assets to be pushed up. This causes higher CAPM beta stocks to be overpriced and the lower CAPM beta stocks to be underpriced, which over time means negative alpha for high CAPM beta stocks and positive alpha for low CAPM beta stocks.

While I believe there is something to this reasoning, it’s not enough to explain the persistence of the volatility effect, and it is not specific. That is, specifically, how much risk seeking results in how much higher or lower alpha?

Meanwhile, there are a lot of smart people in Wall Street who know how to take advantage of pricing anomalies. Yet the volatility effect, which for our purposes has not been defined yet, persists long after it has been documented.

In my view the muti-moment models contribute some insight, but don’t solve anything the way we would like to see.

There is a book that offers a development and discussion of multi-moment models. It’s not for the mathematically faint of heart, but it’s easily found in a PDF download.

Multi-Moment Asset Allocation and Pricing Models, (edited by Emmanuel Jurczenko and Bertrand Maillet, 2006, John Wiley & Sons).

Alpha Found Video 7

Alternative Models – Multi-factor Models

Welcome back to part 7.

In one view, the higher moment models to some degree try to rescue efficient markets. Investors are pricing more than the first two moments. Since CAPM prices assets based on the first two moments, the appearance of inefficiency is only a mismatch between two-moment models and higher-moment models. And this is why CAPM does not work as well as we would like. Meaning markets would be measured as efficient if measured through the lens of higher moments. That's one view.

A multi-factor model makes the case that either CAPM is fundamentally incomplete or markets are inefficient. Whichever it may be, adding additional factors might remedy the incompleteness or inefficiency. By considering not just market returns and covariances in the model, but other systematic factors that influence returns, we could get a better understanding of market dynamics.

To cut to the chase on this, let me say that there are two main multi-factor models that have had the attention of investors and academics alike.

One is the Fama-French three-factor model, and the other is the Fama-French-Carhart four-factor model.

The Fama-French three-factor model has the following three factors:

1. The same first factor as CAPM, which is the excess return of the market, (or the return of a broad market index) scaled by beta. So, all is good, nothing much new here in the first factor.

2. The remaining two factors try to capture two apparent inefficiencies (or perhaps deficiencies, depending on your perspective). One is the small stock effect, which is the observation that small company stocks outperform large company stocks on a risk adjusted basis over long periods of time. This factor is captured statistically by a factor called SMB or small minus big. Just as with the first factor, the sensitivity that a stock or portfolio has to SMB is scaled by its own beta.

3. The third factor is a value factor as measured by book to market ratio, and is called HML, or high minus low. High book to market value minus low book to market value. This factor also gets scaled by its own beta.

The risk-free rate is included, and also an error term for each time period.

The different betas are estimated by ordinary least squares regression. Because of the other two factors, the first beta that scales market returns will not have the same estimated value as it would in the single factor CAPM. With the multi-factor model some of the other factors do the heavy lifting that was previously accomplished by one factor.

It looks like this:

$$R'_{at} = r_f + \alpha'_a + \beta'_a\left(R'_{Mt} - r_f\right) + \beta'_{aSMB}(SMB_t) + \beta'_{aHML}(HML_t) + \varepsilon'_{at} \tag{7.1}$$

The Fama-French-Carhart four-factor model has the same first three factors as the three-factor model, with the addition of a momentum factor.

These models have much higher predictive accuracy than CAPM. At first glance, with this higher accuracy, it would seem the difficulties are solved or mostly solved and we can move on to other problems. However, there are a few of relevant criticisms.

The first criticism is that

1. Multi-factor models are statistical or empirical models, not economic models.

The inputs are driven by data, if the data were different, the model would be different. CAPM was developed by economic reasoning. The data to test it came later.

The three-factor and four-factor models are “fitted” models. The data determine what the model is.

2. Factor models continue to be invented, because to date, none of them sufficiently satisfy investors.

There has been a proliferation of multi-factor models. Even Fama and French have updated their model to a five-factor model, including a profitability factor (RMW - robust minus weak) and an investment factor (CMA - conservative minus aggressive).

The fact that there are several multi-factor models out there indicates indirectly that there is dissatisfaction with each particular multi-factor model. Otherwise, no one would go through the effort to create a new one.

Some other multi-factor models include low volatility as a factor. Which this series of videos will eventually address.

3. Adding factors increases the likelihood of multicollinearity,

where one factor can be estimated by another factor. Adding one new factor may eliminate the need for a previous factor, or perhaps two new factors together can do the work that three others used to do.

Whatever is going on, multicollinearity means that one factor can seem to predict partially the outcome of another. When this is the case, what the underlying contributing factor really is remains an open question.

In defense of multi-factor models, the apparent inefficiencies that the multi-factor models account for might not be inefficiencies. There could easily be hidden risk that systematic risk does not price properly, but gets priced in things like SMB or HML. If this is true then the small cap effect and value anomaly are not anomalies. Their risk is just incompletely accounted for by traditional systematic risk.

For instance, small companies tend to have a higher cost of capital than large companies. This is an increased risk which gets priced and leads to higher alpha as measured by traditional systematic risk alone, but would not lead to higher alpha if risk were properly measured.

Finally,

4. Multi-factor models quantify return anomalies more than explain them.

In the final analysis, I find that the factors of multi-factor models are more descriptive than explanatory. That is, the factors acknowledge that there are mysteries. The value effect, the small stock effect, and other so-called anomalies are mysteries that get quantified to some degree in the multi-factor models. I don’t see that the factors are explained such that we understand them better, they are only quantified better.

Of course, others may see it differently.

Alpha Found Video 8

Explanations Nearby to CAPM

Welcome back to Alpha Found part 8.

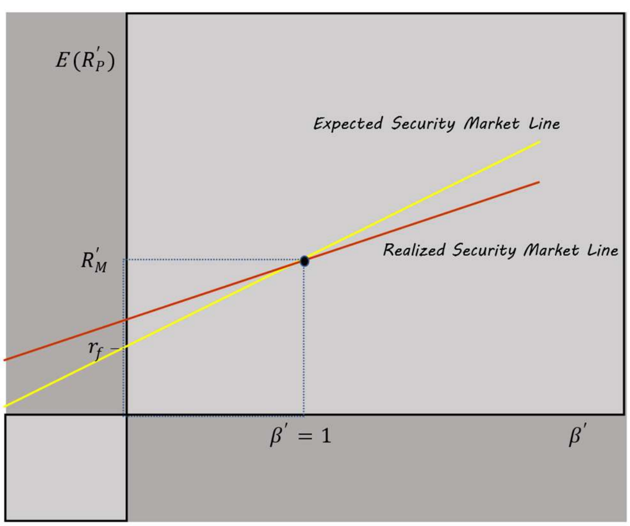
If the multi-moment models don't give us the answer we are looking for, and multi-factor models are more descriptive than explanatory, there is only one place left to go to explain the volatility effect that has been taken seriously. In this case it isn't one idea, but a collection of ideas: those being that the assumptions of CAPM are not wholly valid. When more realistic assumptions are considered, the volatility effect appears as a natural consequence.

Rather than dig through several articles to examine what happens when the assumptions are relaxed, I want to look at just one titled *Explanations for the Volatility Effect: An Overview Based on the CAPM Assumptions*, by David Blitz, Eric Falkenstein, and Pim van Vliet, (Journal of Portfolio Management, Volume 40 No. 3, Spring 2014, Pgs. 61-76). They do an excellent job so it is much better just to go through what they have done rather than try to recreate it.

Their goal is to organize the different possibilities according to which CAPM assumption it violates. My goal is to cover their work enough to give the flavor, rather than dig up all the details. Some points are skipped altogether.

They begin by citing Fama-McBeth [1973], and then Haugen and Heins [1975], who concluded that the risk-return relationship is flatter than CAPM predicts. This is just another way of saying that positive alpha is associated with low CAPM beta. It means the security market line has a lesser slope than predicted.

Figure 8.1
Realized Security Market Line



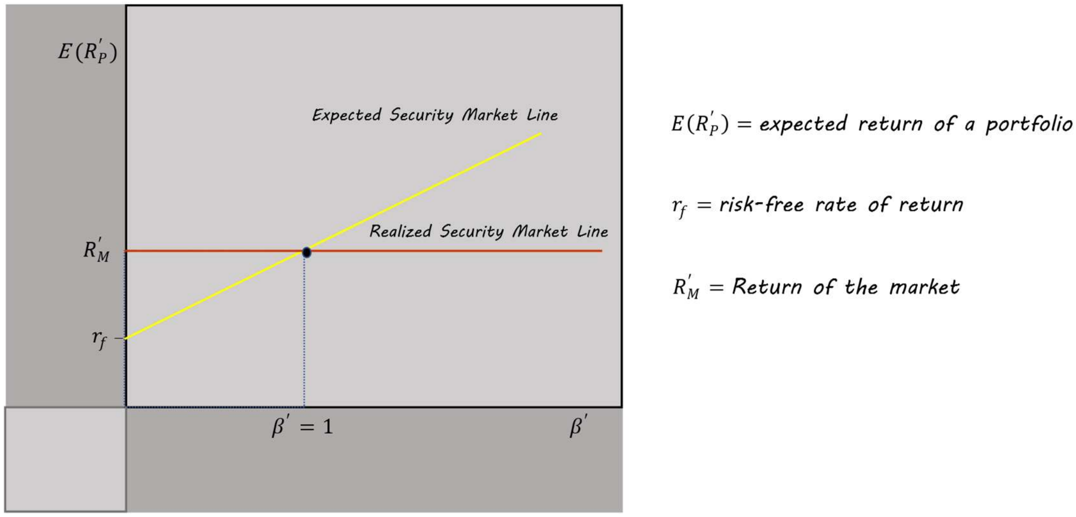
$E(R_P')$ = *expected return of a portfolio*

r_f = *risk-free rate of return*

R_M' = *Return of the market*

Then they cite Fama and French [1992], who note that after adjusting for size effects, systematic risk and return are not connected. Which means there is no security market line. Using size as an additional factor sends us back to multifactor models.

Figure 8.2
Alternate Realized Security Market Line



In some studies, the risk-return relationship is negative, meaning lower risk leads not just to higher alpha, but to higher absolute returns than that achieved by higher CAPM beta investments, which means the security market line slopes downward. These effects exist in international markets as well.

Whatever is going on, it is persistent across time and markets. You can look up the article to get the citations.

Blitz, Falkenstein, and Van Vliet, organize the CAPM assumptions a little differently from Bodie, Kane, and Marcus as listed in the fourth video, but the substance is the same. We're going to look in a superficial way.

The first assumption they relax is "no constraints on leverage," meaning an investor can borrow or sell short without limitation.

In reality there are constraints on both borrowing and short selling. As a consequence, to increase expected return there is no place to go except higher CAPM beta securities, which have higher systematic risk. This increases the demand and therefore increases the price of those securities above fair value, which lowers alpha. This leaves low CAPM beta securities with higher alpha. With no short selling, the optimistic investors for each security become the owners, without offsetting downward price pressure that short sellers would provide. This distorts the risk-return payoff.

For both of these violations of the assumptions, it means investment managers are not completely rational with respect to asset pricing, but may be rational with respect to their personal financial compensation, which is often tied to relative investment performance. That is, analysts can see the overpriced state, but are forced by their inability to use leverage or short selling into buying higher CAPM beta stocks. Perhaps buying high-risk stocks is rational for an investment manager, but not rational for an outside observer who does not have the same pressures. According to the article, constraints on leverage filters into most of the other possible explanations for the volatility effect that arise from violations of the CAPM assumptions.

Next, they look at investor utility, meaning an assessment of what the returns are from an investment in terms of utility rather than in dollars. The thinking goes like this: according to CAPM, investors are utility maximizers.

In reality, investors are concerned with relative utility, not maximum utility. That is, most people want to do better than their neighbors, not necessarily better against an abstract standard. Utility maximizers are motivated by greed whereas those concerned with relative utility are motivated by envy.

This subtle difference has important implications. Relative utility may cause investment managers to seek higher risk investments though their own analysis tells them it is not a great idea after adjusting for risk. Compensation of investment managers motivates risky behavior in that often managers are paid a bonus for outperformance, but no negative bonus for underperformance. The only reliable path to outperformance given the constraints of investment policy is to seek higher risk investments. This increases demand and therefore the price of high CAPM beta securities. There is no downside for the manager.

As an example, investment managers attract clients based on relative performance. If the market is down 10% and clients of the investment manager are down 8%, the clients have lost an absolute value of 8%, but gained 2% in relative performance. If markets are up 20% but the manager is up only 18%, the 2% relative underperformance hurts the managers ability to attract business and may in fact lose business, though the clients had great absolute returns.

CAPM assumes investors are mean-variance optimizers.

In reality investors might have a preference for skewness. Preference for skewness distorts the construction of CAPM. It is a risk seeking behavior that seeks out investments that have potential for outsized returns, thus pushing prices higher than their fair value, leading to lower alpha. This is more likely for high CAPM beta securities rather than low CAPM beta securities. The thought of outsized returns changes investor motivations. If investors are not risk averse, then naturally the investment payoffs will reflect the higher prices paid for higher risk.

The last set of explanations revolve around behavioral finance explanations. There is a rich literature on behavioral finance, to the point that Daniel Kahneman and Richard Thaler have been awarded Nobel Prizes for their work, in 2002 and 2017 respectively.

CAPM assumes investors are rational.

In reality investors make several cognitive and behavioral errors. The representativeness heuristic, mental accounting, overconfidence, among other cognitive and behavioral errors cause unintentional errors in the assessment of value. Specifically, investors can become blinded to the reality of risk, or having seen it, pursue it against their own better judgement.

To talk this through a little, the representativeness heuristic says that “people rely more on appealing anecdotes than on dull statistics.” For example, when one stock that you perhaps haven’t heard about is compared favorably to a stock you have heard about, or when a stock has some similar attributes to a stock or stocks that do well, the reputation of the stock or stocks you have heard about gets mentally transferred to the stock you have not heard about. For these stocks, the price is bid up too high.

Another possibility is mental accounting. Many investors have a portion of their assets set aside for rational investing and another portion set aside for high risk opportunities. This is similar to skewness preference, and also is similar to gambling behavior in that people will gamble with a maximum loss in mind, and will continue to gamble until that limit is reached. The rest of their money is not to be gambled with.

The final behavioral explanation is overconfidence. Plenty of investors think they have above average abilities and can take higher risk without the associated downside cost. These investors are optimistic, which is a good trait under most circumstances, but for investing it results in risk seeking behavior. Confident investors and market timers do not play in the low-risk end of the sand box, which again means higher risk stocks get more attention and higher prices than they deserve. Hence, the volatility effect.

The paper reviewed here has more to it than this brief review would indicate. I find the reasons and reasoning to be compelling. However, it is frustratingly non-specific. How much of which cause results in how much alpha for low CAPM beta securities? I suspect the authors would like to have the same information.

There have been a large number of proposed explanations for the volatility effect, some of which I have not mentioned. The problem feels unresolved in an annoying kind of way.

The next two videos set the table, then we begin looking at the new model.

Alpha Found Video 9

Setting the Table Part 1: Insights on Correlation and Sharpe Ratios

Welcome back to part 9 of Alpha Found.

With this video we begin new material. There are many thoughts that I would like to express all at once so that questions and objections can be answered before they're asked. Since this is not possible, the answers will have to come in their own time.

The mathematics will not get beyond elementary statistics; however, from this point forward I'm assuming the viewer can keep up without much trouble.

This video has insights that to my knowledge are new, and other insights that are old, but I'll derive them carefully anyway.

The old insights relate to the fact that correlation and Sharpe ratios would seem to be related.

The new insights are that beta and correlation affect one another in a structured way, which then affects what is expected in the relationship between alphas and Sharpe ratios.

A simple method of portfolio construction is used throughout the videos that can be analyzed algebraically by standard ordinary least squares (OLS) estimation metrics and Sharpe Ratios without regard to data. Data is examined in this video and in some later videos to show how it is consistent with the algebraic analysis.

For our purposes, portfolio construction occurs by ranking all stocks in a capitalization weighted index by ordinary least squares estimates of CAPM β' with alpha from highest to lowest, and using that index as a standard for comparison.

The index is divided into two portfolios, each with 50% of the market capitalization, one called the high CAPM $\hat{\beta}'$ portfolio and the other one called the low CAPM $\hat{\beta}'$ portfolio. The two portfolios are mutually exclusive and exhaustive of the index. Portfolio weights are rebalanced each time period.

As I said previously, the prime notation is used whenever a statistic or parameter has its ordinary meaning in the context of CAPM. Statistics or parameters without a prime mark are defined when they are used for the first time. Here I will just point out the pattern.

Statistics and parameters with the subscript H refer to the relevant measure for the high CAPM β' portfolio, and the subscript L refers to the relevant measure for the low CAPM β' portfolio. The subscript M refers to the market (or index) portfolio, which is the equal weight sum of the high and low CAPM β' portfolios. Take a moment to look through the notation to see how it is used.

Notation for the high CAPM $\hat{\beta}'$ portfolio

$\hat{\beta}'_H$ = Estimated CAPM beta for the high CAPM beta portfolio

$\hat{\alpha}'_H$ = Estimated alpha for the high CAPM beta portfolio

$\hat{\sigma}'^2_H$ = Estimated variance of the high CAPM beta portfolio

Notation for the market portfolio

$\hat{\sigma}'^2_M$ = Estimated variance of the market portfolio

Notation for the low CAPM $\hat{\boldsymbol{\beta}}'$ portfolio

$\hat{\beta}'_L$ = Estimated CAPM beta for the low CAPM beta portfolio

$\hat{\alpha}'_L$ = Estimated alpha for the low CAPM beta portfolio

$\hat{\sigma}'^2_L$ = Estimated variance of the low CAPM beta portfolio

The phrase *generally rising market* is any series of time periods where both the arithmetic and geometric mean returns are positive. A *generally flat market* occurs when the arithmetic mean return is positive while the geometric mean return is negative, and a *generally falling market* is a series of time periods when both the arithmetic and geometric mean returns are negative.

By construction, an OLS regression of the two portfolios described above will have estimated CAPM $\hat{\beta}'$s of their two portfolios summing to two, with an average of one.

$$\hat{\beta}'_L + \hat{\beta}'_H = 2 \tag{9.1}$$

I should mention that portfolios constructed in this way will give results that are heavily biased or less than rigorous. This is one of the problems with using OLS to estimate CAPM beta for a portfolio consisting of a large portion of an index. The new model, when we get to it, will not have this problem.

Also, the regression residuals for each observation of the two portfolios are equal in magnitude but opposite in sign. Consequently, their residual variances (idiosyncratic risks) are identical, so for simplicity it is referred to as $\hat{\sigma}'^2_\varepsilon$ for both portfolios.

$$\hat{\sigma}'^2_{\varepsilon H} = \hat{\sigma}'^2_{\varepsilon L} = \hat{\sigma}'^2_\varepsilon \tag{9.2}$$

Their realized $\hat{\alpha}'$s, if they exist, are equal in magnitude, opposite in sign, and sum to zero.

As shown here:

$$\hat{\alpha}'_L = -\hat{\alpha}'_H \tag{9.3}$$

Under this construction, in the standard CAPM framework including $\hat{\alpha}'$, the $E(\hat{\alpha}'_L)$ is the same as the $E(\hat{\alpha}'_H)$, each being zero.

$$E(\hat{\alpha}'_L) = E(\hat{\alpha}'_H) = 0 \tag{9.4}$$

Also, assuming complete diversification or efficiency of the two portfolios, the $E(\hat{S}'_L)$, $E(\hat{S}'_H)$ and $E(\hat{S}'_M)$ is the same.

$$E(\hat{S}'_L) = E(\hat{S}'_H) = E(\hat{S}'_M) \tag{9.5}$$

I should mention that some might say if the two portfolios were completely diversified there would be no residuals. Ex-ante that may be true, but given random variation, it would not be true ex-post. As a consequence, it is impossible for (9.4) and (9.5) to be realized at the same time. See the discussion later in this video and in later videos for details.

Each of (9.4) and (9.5) is a possible interpretation of what it means for markets to be efficient. The expectations found in (9.4) and (9.5) are not realized in actual market data. Instead, results from ex-post portfolios so constructed usually have the following pattern:

$$\hat{\alpha}'_L > 0 > \hat{\alpha}'_H \tag{9.6}$$

$$\hat{S}'_L > \hat{S}'_M > \hat{S}'_H \tag{9.7}$$

In light of the expectations of (9.4) and (9.5), the commonly realized statistical relationships found in (9.6) and (9.7) will serve as the definition of the volatility effect. To the extent that (9.6) and (9.7) can be expressed as mathematical expectations (ex-ante) the volatility effect is resolved.

With the relationships in (9.1) – (9.3), three interesting observations follow:

$$\hat{\beta}'_L + \hat{\beta}'_H = 2 \tag{9.1}$$

$$\hat{\sigma}'^2_{\varepsilon H} = \hat{\sigma}'^2_{\varepsilon L} = \hat{\sigma}'^2_{\varepsilon} \tag{9.2}$$

$$\hat{\alpha}'_L = -\hat{\alpha}'_H \tag{9.3}$$

The first observation is that with equal idiosyncratic risk for each portfolio, the correlation of the high CAPM $\hat{\beta}'$ portfolio is necessarily higher than the correlation of the low CAPM $\hat{\beta}'$ portfolio.

$$\hat{\rho}'_{H,M} > \hat{\rho}'_{L,M} \qquad \leftrightarrow \qquad \hat{\rho}'_{H,M} - \hat{\rho}'_{L,M} > 0 \tag{9.8}$$

Let me prove this.

Assuming that in the usual case $0 < \hat{\rho}' < 1$, then from any textbook on elementary statistics:

$$\hat{\rho}' = \sqrt{R^2} = \sqrt{\frac{SSM}{SST}} = \sqrt{\frac{\Sigma(\hat{Y}_t - \bar{Y})^2}{\Sigma(Y_t - \bar{Y})^2}}$$

Moving the right-hand term into the context of CAPM with these substitutions:

$$\hat{Y}_t = \hat{\alpha}' + \hat{\beta}'(R'_{Mt} - r_f) \qquad \bar{Y} = \hat{\alpha}' + \hat{\beta}'(\bar{R}'_M - r_f) \qquad Y_t = \hat{\alpha}' + \hat{\beta}'(R'_{Mt} - r_f) + \hat{\varepsilon}'_t$$

gives this messy term on the left:

$$\sqrt{\frac{\Sigma\left(\left(\hat{\alpha}' + \hat{\beta}'(R'_{Mt} - r_f)\right) - \left(\hat{\alpha}' + \hat{\beta}'(\bar{R}'_M - r_f)\right)\right)^2}{\Sigma\left(\left(\hat{\alpha}' + \hat{\beta}'(R'_{Mt} - r_f) + \hat{\varepsilon}'_t\right) - \left(\hat{\alpha}' + \hat{\beta}'(\bar{R}'_M - r_f)\right)\right)^2}} = \sqrt{\frac{\Sigma\left(\left(\hat{\beta}'(R'_{Mt} - r_f)\right) - \left(\hat{\beta}'(\bar{R}'_M - r_f)\right)\right)^2}{\Sigma\left(\left(\hat{\beta}'(R'_{Mt} - r_f) + \hat{\varepsilon}'_t\right) - \left(\hat{\beta}'(\bar{R}'_M - r_f)\right)\right)^2}}$$

The alpha terms on the left-hand side cancel and so don't appear on the right-hand side. Then in the next step, $\hat{\beta}'$ is pulled out of the parenthesis and gets squared in the process.

$$\sqrt{\frac{\hat{\beta}'^2\,\Sigma\left((R'_{Mt}-r_f)-(\bar{R}'_M-r_f)\right)^2}{\hat{\beta}'^2\,\Sigma\left((R'_{Mt}-r_f)+\frac{\hat{\varepsilon}'_t}{\hat{\beta}'}-(\bar{R}'_M-r_f)\right)^2}} = \hat{\rho}' = \sqrt{\frac{\Sigma\left((R'_{Mt}-r_f)-(\bar{R}'_M-r_f)\right)^2}{\Sigma\left((R'_{Mt}-r_f)+\frac{\hat{\varepsilon}'_t}{\hat{\beta}'}-(\bar{R}'_M-r_f)\right)^2}} \tag{9.9}$$

The $\hat{\beta}'s$ cancel out in the right-hand term.

In the denominator of (9.9), if $\hat{\beta}'$ is larger than 1, the contribution of the residuals to the denominator is diminished, therefore increasing correlation. If $\hat{\beta}'$ is less than 1, the contribution of the residuals to the denominator is magnified, therefore reducing correlation. Consequently, when an index is divided equally between two mutually exclusive and exhaustive portfolios, as long as there are regression residuals, the one with the lower CAPM $\hat{\beta}'$ will always have the lower correlation.

Which means observation #1 is always true.

$$\hat{\rho}'_{H,M} > \hat{\rho}'_{L,M} \quad \leftrightarrow \quad \hat{\rho}'_{H,M} - \hat{\rho}'_{L,M} > 0 \tag{9.8}$$

If an index is divided into more than two portfolios (for example, deciles) the regression residuals will no longer be equal in magnitude and correlation will vary more freely. However, there is a general tendency for correlation to follow CAPM $\hat{\beta}'$ when an index is broken into a larger number of portfolios, but it shows up in an unexpected way.

If we look at the data from *The Volatility Effect: Lower Risk without Lower Return*, by David Blitz, and Pim van Vliet, (Journal of Portfolio Management, Volume 34 No. 1, Fall 2007, Pgs. 102-113) they report CAPM betas ranked by decile along with the standard deviation of each CAPM beta decile, and the market standard deviation. From this, the correlation each decile has with the market can be easily derived.

Blitz and Van Vliet provide information on markets in the US, Japan, and Europe, over the time period 1986 through 2006.

If we start with the formula for CAPM beta:

$$\hat{\beta}'_a = \hat{\rho}'_{a,M} \frac{\hat{\sigma}'_a}{\hat{\sigma}'_M} \tag{6.1}$$

By moving terms around, the correlations for each beta decile can be calculated using this formula:

$$\hat{\rho}'_{a,M} = \frac{\hat{\beta}'_a \hat{\sigma}'_M}{\hat{\sigma}'_a} \tag{9.10}$$

Here is the result of the correlation calculation. It's in the fourth line of each of three statistical groupings found in Table 9.1.

Table 9.1
Monthly Data from *The Volatility Effect: Lower Risk Without Lower Return*
by David Blitz and Pim Van Vliet (2007)

US Market Volatility Statistics Ranked by Beta Deciles 1986 - 2006

	1	2	3	4	5	6	7	8	9	10
Beta	0.45	0.70	0.82	0.86	0.94	0.95	1.01	1.16	1.39	1.77
Standard Deviation	0.1200	0.1370	0.1540	0.1590	0.1700	0.1670	0.1800	0.2070	0.2580	0.3650
Market Std. Dev.	0.1710	0.1710	0.1710	0.1710	0.1710	0.1710	0.1710	0.1710	0.1710	0.1710
Correlation	0.6413	0.8737	0.9105	0.9249	0.9455	0.9728	0.9595	0.9583	0.9213	0.8292

Europe Market Volatility Statistics Ranked by Beta Deciles 1986 - 2006

	1	2	3	4	5	6	7	8	9	10
Beta	0.64	0.74	0.82	0.93	0.94	0.98	1.06	1.12	1.29	1.49
Standard Deviation	0.1240	0.1420	0.1510	0.1690	0.1720	0.1780	0.1900	0.2020	0.2370	0.2870
Market Std. Dev.	0.1750	0.1750	0.1750	0.1750	0.1750	0.1750	0.1750	0.1750	0.1750	0.1750
Correlation	0.9032	0.9120	0.9503	0.9630	0.9564	0.9635	0.9763	0.9703	0.9525	0.9085

Japan Market Volatility Statistics Ranked by Beta Deciles 1986 - 2006

	1	2	3	4	5	6	7	8	9	10
Beta	0.61	0.78	0.87	0.91	0.98	1.02	1.06	1.15	1.21	1.42
Standard Deviation	0.1520	0.1800	0.1960	0.2010	0.2150	0.2230	0.2320	0.2530	0.2710	0.3300
Market Std. Dev.	0.2150	0.2150	0.2150	0.2150	0.2150	0.2150	0.2150	0.2150	0.2150	0.2150
Correlation	0.8628	0.9317	0.9543	0.9734	0.9800	0.9834	0.9823	0.9773	0.9600	0.9252

At first glance the correlation data may not show much, but on a closer look, if we examine the correlation of each decile, and its corresponding symmetric decile, in each case the higher beta decile has a higher correlation than the lower beta decile, as seen here in Table 9.2. (See next page).

You may have to stop the video from time to time to review the tables.

In total there are 15 pairs of matched deciles, and each of 15 times the higher beta decile has a higher correlation than the symmetric lower beta decile.

Using the same procedure on Black, Jensen, and Scholes, Table 9.3 shows there are four time periods with ten deciles shown grouped at the bottom, plus the aggregate time period shown in the first grouping.

As shown in Table 9.4, in four of the five time periods there is the same pattern of higher correlation with the higher beta decile when matched against the symmetric lower beta decile. In one time period 4 of 5 matched deciles have the same pattern.

Table 9.2
Correlations Derived from *The Volatiltiy Effect: Lower Risk Without Lower Return* by David Blitz and Pim Van Vliet (2007)

US Market Correlation Statistics Compared by Matched CAPM Beta Deciles 1986 - 2006

Matched Correlation Deciles	Correlations	Difference	
6th vs 5th	0.9728 - 0.9455 =	0.02723	In 5 of 5 cases the correlation of the
7th vs 4th	0.9595 - 0.9249 =	0.03459	higher β' decile is higher than the
8th vs 3rd	0.9583 - 0.9105 =	0.04774	correlation of the symmetric decile.
9th vs 2nd	0.9213 - 0.8737 =	0.04756	
10th vs 1st	0.8292 - 0.6413 =	0.18798	

European Market Correlation Statistics Compared by Matched CAPM Beta Deciles 1986 - 2006

Matched Correlation Deciles	Correlations	Difference	
6th vs 5th	0.9635 - 0.9564 =	0.00709	In 5 of 5 cases the correlation of the
7th vs 4th	0.9763 - 0.9630 =	0.01330	higher β' decile is higher than the
8th vs 3rd	0.9703 - 0.9503 =	0.01997	correlation of the symmetric decile.
9th vs 2nd	0.9525 - 0.9120 =	0.04056	
10th vs 1st	0.9085 - 0.9032 =	0.00531	

Japanese Market Correlation Statistics Compared by Matched CAPM Beta Deciles 1986 - 2006

Matched Correlation Deciles	Correlations	Difference	
6th vs 5th	0.9834 - 0.9800 =	0.00341	In 5 of 5 cases the correlation of the
7th vs 4th	0.9823 - 0.9734 =	0.00894	higher β' decile is higher than the
8th vs 3rd	0.9773 - 0.9543 =	0.02294	correlation of the symmetric decile.
9th vs 2nd	0.9600 - 0.9317 =	0.02830	
10th vs 1st	0.9252 - 0.8628 =	0.06232	

Since the relationship has been proved to be necessarily true when an index is divided into two halves, it seems that for deciles the pattern is far too consistent to be just a happy coincidence. This issue is pursued further in videos 30 and 31.

A few minutes ago, I said there were three observations that flow directly from equations (9.1) – (9.3). The relationship between correlations was the first. The second observation is that the Sharpe ratio of any individual security or portfolio with zero $\hat{\alpha}'$ equals the correlation multiplied by the market Sharpe ratio.

$$\hat{S}_L' = \hat{\rho}_{L,M}'\hat{S}_M' \tag{9.11a}$$

$$\hat{S}_H' = \hat{\rho}_{H,M}'\hat{S}_M' \tag{9.11b}$$

Table 9.3
Monthly Data from *The Capital Asset Pricing Model: Some Empirical Tests*
by Fischer Black, Michael C. Jensen, and Myron Scholes (1972)
Volatility Statistics Ranked by Beta Deciles 1931 - 1965

Portfolio Beta Ranked by Deciles for Four Sub-periods

	1	2	3	4	5	6	7	8	9	10
Beta Subperiod 1	0.4843	0.6222	0.7510	0.8569	0.9197	1.0750	1.1813	1.2620	1.3993	1.5416
Beta Subperiod 2	0.5626	0.6647	0.7675	0.8114	0.9254	0.9697	1.0861	1.1938	1.3196	1.7157
Beta Subperiod 3	0.4868	0.6547	0.7714	0.9180	0.9851	1.0474	1.1216	1.1822	1.3598	1.5427
Beta Subperiod 4	0.6226	0.6614	0.7800	0.8601	0.9248	0.9957	1.0655	1.1818	1.2764	1.4423
Across all Periods	0.4992	0.6291	0.7534	0.8531	0.9229	1.0572	1.1625	1.2483	1.3838	1.5614

Portfolio Standard Deviations Ranked by Beta Deciles for Four Sub-periods

	1	2	3	4	5	6	7	8	9	10
Standard Deviation 1	0.0850	0.1024	0.1211	0.1377	0.1484	0.1715	0.1886	0.2023	0.2243	0.2504
Standard Deviation 2	0.0392	0.0441	0.0494	0.0519	0.0586	0.0618	0.0690	0.0758	0.0841	0.1187
Standard Deviation 3	0.0203	0.0253	0.0289	0.0340	0.0364	0.0385	0.0413	0.0436	0.0505	0.0581
Standard Deviation 4	0.0265	0.0277	0.0312	0.0340	0.0365	0.0391	0.0420	0.0463	0.0503	0.0577
Across all Periods	0.0495	0.0586	0.0685	0.0772	0.0836	0.0950	0.1045	0.1126	0.1248	0.1445

Market Standard Deviations for Four Sub-periods

	1	2	3	4	5	6	7	8	9	10
Market Std. Dev. 1	0.1587	0.1587	0.1587	0.1587	0.1587	0.1587	0.1587	0.1587	0.1587	0.1587
Market Std. Dev. 2	0.0624	0.0624	0.0624	0.0624	0.0624	0.0624	0.0624	0.0624	0.0624	0.0624
Market Std. Dev. 3	0.0363	0.0363	0.0363	0.0363	0.0363	0.0363	0.0363	0.0363	0.0363	0.0363
Market Std. Dev. 4	0.0386	0.0386	0.0386	0.0386	0.0386	0.0386	0.0386	0.0386	0.0386	0.0386
Across all Periods	0.0891	0.0891	0.0891	0.0891	0.0891	0.0891	0.0891	0.0891	0.0891	0.0891

Correlations Ranked by Beta Decile for Four Sub-periods

	1	2	3	4	5	6	7	8	9	10
Correlations 1	0.9042	0.9643	0.9842	0.9876	0.9835	0.9948	0.9940	0.9900	0.9901	0.9770
Correlations 2	0.8956	0.9405	0.9695	0.9756	0.9854	0.9791	0.9822	0.9828	0.9791	0.9019
Correlations 3	0.8705	0.9394	0.9689	0.9801	0.9824	0.9875	0.9858	0.9843	0.9774	0.9639
Correlations 4	0.9069	0.9217	0.9650	0.9765	0.9780	0.9830	0.9792	0.9853	0.9795	0.9649
Across all Periods	0.8981	0.9560	0.9793	0.9851	0.9833	0.9915	0.9914	0.9882	0.9875	0.9625

Although observation #2 is not new, we should prove it anyway. In the process of proving observation #2, most of observation #3 will also be proved.

Proof:

Here is the formula for the Sharpe ratio:

$$\hat{S}'_a = \frac{\bar{R}'_M - r_f}{\hat{\sigma}'_a} \tag{9.12}$$

Table 9.4
Correlations Derived from *The Capital Asset Pricing Model: Some Empirical Tests* by Fischer Black, Michael C. Jensen, and Myron Scholes (1972) aka BJS

BJS Correlation Statistics Compared by Matched Beta Deciles 1931 - 1965

Matched Corrleation Deciles	Correlations	Difference	
6th vs 5th	0.9915 - 0.9833 =	0.00820	In 5 of 5 cases the correlation of the higher β decile is
7th vs 4th	0.9914 - 0.9851 =	0.00630	higher than the correlation of the symmetric decile.
8th vs 3rd	0.9882 - 0.9793 =	0.00890	
9th vs 2nd	0.9875 - 0.9560 =	0.03150	
10th vs 1st	0.9625 - 0.8981 =	0.06440	

BJS Correlation Statistics Compared by Matched Beta Deciles Subperiod 1

Matched Corrleation Deciles	Correlations	Difference	
6th vs 5th	0.9948 - 0.9835 =	0.01123	In 5 of 5 cases the correlation of the higher β decile is
7th vs 4th	0.9940 - 0.9876 =	0.00644	higher than the correlation of the symmetric decile.
8th vs 3rd	0.9900 - 0.9842 =	0.00584	
9th vs 2nd	0.9901 - 0.9643 =	0.02576	
10th vs 1st	0.9042 - 0.9042 =	0.07283	

BJS Correlation Statistics Compared by Matched Beta Deciles Subperiod 2

Matched Corrleation Deciles	Correlations	Difference	
6th vs 5th	0.9791 - 0.9854 =	-0.00629	In 4 of 5 cases the correlation of the higher β decile is
7th vs 4th	0.9822 - 0.9756 =	0.00666	higher than the correlation of the symmetric decile.
8th vs 3rd	0.9828 - 0.9695 =	0.01329	It would not be surprising to find the reported β or
9th vs 2nd	0.9791 - 0.9405 =	0.03858	standard deviation of decile 5 or 6 is an error, given
10th vs 1st	0.9019 - 0.8956 =	0.00637	the difference in correlations is negative.

BJS Correlation Statistics Compared by Matched Beta Deciles Subperiod 3

Matched Corrleation Deciles	Correlations	Difference	
6th vs 5th	0.9875 - 0.9824 =	0.00515	In 5 of 5 cases the correlation of the higher β decile is
7th vs 4th	0.9858 - 0.9801 =	0.00571	higher than the correlation of the symmetric decile.
8th vs 3rd	0.9843 - 0.9689 =	0.01534	
9th vs 2nd	0.9774 - 0.9394 =	0.03809	
10th vs 1st	0.9639 - 0.8705 =	0.09337	

BJS Correlation Statistics Compared by Matched Beta Deciles Subperiod 4

Matched Corrleation Deciles	Correlations	Difference	
6th vs 5th	0.9915 - 0.9833 =	0.00820	In 5 of 5 cases the correlation of the higher β decile is
7th vs 4th	0.9914 - 0.9851 =	0.00630	higher than the correlation of the symmetric decile.
8th vs 3rd	0.9882 - 0.9793 =	0.00890	
9th vs 2nd	0.9875 - 0.9560 =	0.03150	
10th vs 1st	0.9625 - 0.8981 =	0.06440	

Beginning with (9.8),

$$\hat{\rho}'_{H,M} - \hat{\rho}'_{L,M} > 0 \tag{9.8}$$

If $\sigma'_M > 0$, and with a generally rising market, the difference in correlations can be multiplied by the market Sharpe ratio to get the following:

$$\left(\hat{\rho}'_{H,M} - \hat{\rho}'_{L,M}\right)\hat{S}'_M = \left(\hat{\rho}'_{H,M} - \hat{\rho}'_{L,M}\right)\frac{\bar{R}'_M - r_f}{\hat{\sigma}'_M} > 0$$

Distributing the denominator gives:

$$\left(\frac{\hat{\rho}'_{H,M}}{\hat{\sigma}'_M} - \frac{\hat{\rho}'_{L,M}}{\hat{\sigma}'_M}\right)\left(\bar{R}'_M - r_f\right) > 0$$

And then multiplying each piece in the parenthesis by 1 in the form of portfolio standard deviation divided by portfolio standard deviation:

$$\left(\frac{\hat{\rho}'_{H,M}}{\hat{\sigma}'_M} - \frac{\hat{\rho}'_{L,M}}{\hat{\sigma}'_M}\right)\left(\bar{R}'_M - r_f\right) = \left(\frac{\hat{\rho}'_{H,M}\frac{\hat{\sigma}'_H}{\hat{\sigma}'_M}}{\hat{\sigma}'_H} - \frac{\hat{\rho}'_{L,M}\frac{\hat{\sigma}'_L}{\hat{\sigma}'_M}}{\hat{\sigma}'_L}\right)\left(\bar{R}'_M - r_f\right) > 0$$

Since the formula for CAPM beta is in the numerator of each term on the right, then CAPM beta can replace the numerator of each term:

$$\left(\frac{\hat{\rho}'_{H,M}\frac{\hat{\sigma}'_H}{\hat{\sigma}'_M}}{\hat{\sigma}'_H} - \frac{\hat{\rho}'_{L,M}\frac{\hat{\sigma}'_L}{\hat{\sigma}'_M}}{\hat{\sigma}'_L}\right)\left(\bar{R}'_M - r_f\right) = \left(\frac{\hat{\beta}'_H}{\hat{\sigma}'_H} - \frac{\hat{\beta}'_L}{\hat{\sigma}'_L}\right)\left(\bar{R}'_M - r_f\right) > 0$$

Then multiply through by the average excess return of the market, and then add and subtract the risk-free rate in the numerator of each term.

$$\frac{\hat{\beta}'_H(\bar{R}'_M - r_f)}{\hat{\sigma}'_H} - \frac{\hat{\beta}'_L(\bar{R}'_M - r_f)}{\hat{\sigma}'_L} = \frac{\hat{\beta}'_H(\bar{R}'_M - r_f) + r_f - r_f}{\hat{\sigma}'_H} - \frac{\hat{\beta}'_L(\bar{R}'_M - r_f) + r_f - r_f}{\hat{\sigma}'_L} > 0$$

If we assume $\hat{\alpha}'_L = \hat{\alpha}'_H = 0$, then the Sharpe ratio of the high CAPM beta portfolio is higher than the Sharpe ratio of the low CAPM beta portfolio.

$$\frac{\bar{R}'_M - r_f}{\hat{\sigma}'_H} - \frac{\bar{R}'_M - r_f}{\hat{\sigma}'_L} > 0 \quad \rightarrow \quad S'_H - S'_L > 0$$

$$\hat{S}'_H > \hat{S}'_L \tag{9.13}$$

By inspection you will see that this pattern reverses if there is a generally falling market.

Recognizing that the two portfolios have different correlations, the third important observation follows.

The market portfolio, having a correlation of 1 with itself, will necessarily have a higher correlation than the high CAPM beta portfolio, which will have a higher correlation than the low CAPM beta portfolio.

$$\hat{\rho}'_{M,M} > \hat{\rho}'_{H,M} > \hat{\rho}'_{L,M} \tag{9.14}$$

Therefore, in a context with idiosyncratic risk, if the realized $\hat{\alpha}' = 0$, the relationship between Sharpe ratios is necessarily as shown on the screen.

$$\hat{S}'_M > \hat{S}'_H > \hat{S}'_L \tag{9.15}$$

This is contrary to the observed volatility effect.

The conclusion is, in a generally rising market with idiosyncratic risk, if the Sharpe ratios of the high and low CAPM beta portfolios are equal, the alphas are not zero.

$$\text{If } \hat{S}'_M > \hat{S}'_H = \hat{S}'_L \text{ then } \hat{\alpha}'_L = -\hat{\alpha}'_H \neq 0$$

Putting it together leads to one of three possibilities:

$$\text{If } \hat{\alpha}'_L = \hat{\alpha}'_H = 0, \text{ then } \hat{S}'_M > \hat{S}'_H > \hat{S}'_L \tag{9.16}$$

$$\text{If } \hat{\alpha}'_L > 0 > \hat{\alpha}'_H, \text{ either } \hat{S}'_M > \hat{S}'_H = \hat{S}'_L \tag{9.17}$$

$$\text{or } \hat{S}'_L > \hat{S}'_M > \hat{S}'_H \tag{9.18}$$

In every study I could locate, a decile variant of (9.18) held in market data.

As a preview of coming attractions: the various expectations in (9.16) - (9.18) can be derived mathematically from assumptions about market dynamics without reference to financial, economic, or behavioral issues, although such issues inform market dynamics. When that mathematical derivation is complete, then to that extent the volatility effect is resolved.

That is, (9.16) - (9.18), and several other possibilities will eventually be shown to be mathematical expectations, depending on conditions. We are not quite ready for that, but we're getting close.

In this video it has been shown that correlation is connected to CAPM beta, and that under CAPM expectations with ordinary random variation in the data, the Sharpe ratio of the low CAPM beta portfolio is expected to be lower than the Sharpe ratio of both the market portfolio and the high CAPM beta portfolio. This expectation is not met in real data, and it's not met in a systematic way. This difference between expectation and reality has come to be called the volatility effect.

The next video visits some data. By using the two-portfolio index construction method presented in this video we will see the positive alpha of low CAPM beta portfolios is much more prevalent than previously thought.

Alpha Found Video 10

Setting the Table Part 2: The Volatility Effect Exists at Every Scale

Welcome once more.

In some ways this video is not needed. All it contains is a new set of calculations that drives home the already well-established idea that positive alpha is strongly associated with low CAPM beta.

In the last video I set up the portfolio construction method that will be used in the rest of the videos. The intention is to make things as simple as possible by dividing investment opportunities into two portfolios, one of high CAPM beta stocks and one of low CAPM beta stocks.

The mathematics that describes this construction is the same whether the two portfolios contain 1,000 securities each or just one security each. A financial economist may know the difference, but the mathematical methods do not.

What follows is some data analysis using two-stock indices. The method is not completely rigorous, but in my opinion, it is good enough to make the point and then use as motivation for the new model that will be introduced in the next video.

1. I acquired total return data for stocks from 1994 through 2019.
2. The stocks were separated into two-year intervals such that each stock had monthly return data for the entire two years.

Then,

3. two stocks were randomly selected from the list to form a two-stock index. Each stock was given a 50% weighting in the index. Portfolio weights were rebalanced each month. No transaction costs, taxes, or any other expense or market friction was included

that would make the procedure more realistic. Neither was a risk-free rate included, making these calculations consistent with the market model.

Then,

4. Using this two-stock index as a benchmark, I calculated the usual statistics of mean return, standard deviation, correlation, CAPM beta, alpha, and Sharpe ratio for each of the two stocks and the index itself.

Since the two stocks are randomly selected, they might both be low CAPM beta stocks, high CAPM beta stocks, one of each, or have some other likely or unlikely combination of attributes. The important point is that whatever attributes they may have when compared to a large index, those attributes might change when compared to one another, or not.

5. This procedure was repeated 5,000 times for each two-year period.

Note that for each pair, there are three “portfolios” to observe: the low CAPM beta portfolio, the high CAPM beta portfolio, and the Market portfolio, although they are hardly portfolios since there is only one stock in each portfolio, with two stocks in the “index” portfolio.

According to CAPM and efficient markets, the probability of positive alpha for a given portfolio in this construction is 50%.

Also, remember it was said in the last video that in a generally falling market, the order of expected Sharpe ratios reverses. The sign of expected alpha also reverses in a falling market.

The outcomes of the 5,000 randomly constructed two-stock portfolios were tabulated concerning alpha, CAPM beta, and Sharpe ratios for each time period.

Here are the results:

Table 10.1
Summary Statistics for Random Pairs of Stocks for 14 time periods

	1994-1995	1996-1997	1998-1999	2000-2001	2002-2003	2004-2005	2006-2007	2008-2009	2010-2011	2012-2013	2014-2015	2016-2017	2018-2019	Down
Number of pairs	4,996	4,998	5,000	5,000	4,998	4,998	5,000	4,998	4,999	4,997	4,997	4,999	4,999	4,991
Low Beta-High Alpha	3,332	3,782	2,200	3,148	3,486	3,200	2,846	2,505	3,157	3,744	3,196	3,628	3,261	1,998
% Low Beta-High Alpha	66.69%	75.67%	44.00%	62.96%	69.75%	64.03%	56.92%	50.12%	63.15%	74.92%	63.96%	72.57%	65.23%	40.03%
Average α' for Low β'	0.404%	0.900%	-0.232%	0.466%	0.658%	0.416%	0.208%	-0.026%	0.376%	0.798%	0.348%	0.652%	0.360%	-0.486%
Average Low Beta	0.6372	0.5727	0.6013	0.5505	0.5488	0.5735	0.5898	0.6529	0.6461	0.5627	0.5843	0.5643	0.6015	0.6872
Avg. Market Return	1.26%	2.14%	1.05%	1.40%	2.08%	1.38%	0.71%	0.36%	1.07%	2.40%	0.16%	1.63%	0.77%	-3.96%
Sharpe Ratio order														
Low-Market-High	2,277	2,503	1,636	2,201	2,539	2,282	2,176	2,056	2,514	2,347	2,601	2,497	2,671	1,284
Low-High-Market	37	114	213	101	16	89	180	155	57	26	206	35	91	1,104
Market-Low-High	613	817	229	517	518	459	306	196	361	836	268	671	270	-
Market-High-Low	944	823	431	724	817	827	564	363	552	1,084	439	901	512	7
High-Market-Low	1,103	729	2,351	1,401	1,089	1,308	1,690	2,127	1,474	699	1,385	885	1,406	1,842
High-Low-Market	22	12	140	56	19	33	84	101	41	5	98	10	49	754
Sharpe Ratio order														
% Low-Market-High	45.58%	50.08%	32.72%	44.02%	50.80%	45.66%	43.52%	41.14%	50.29%	46.97%	52.05%	49.95%	53.43%	25.73%
% Low-High-Market	0.74%	2.28%	4.26%	2.02%	0.32%	1.78%	3.60%	3.10%	1.14%	0.52%	4.12%	0.70%	1.82%	22.12%
% Market-Low-High	12.27%	16.35%	4.58%	10.34%	10.36%	9.18%	6.12%	3.92%	7.22%	16.73%	5.36%	13.42%	5.40%	0.00%
% Market-High-Low	18.90%	16.47%	8.62%	14.48%	16.35%	16.55%	11.28%	7.26%	11.04%	21.69%	8.79%	18.02%	10.24%	0.14%
% High-Market-Low	22.08%	14.59%	47.02%	28.02%	21.79%	26.17%	33.80%	42.56%	29.49%	13.99%	27.72%	17.70%	28.13%	36.91%
% High-Low-Market	0.44%	0.24%	2.80%	1.12%	0.38%	0.66%	1.68%	2.02%	0.82%	0.10%	1.96%	0.20%	0.98%	15.11%

The first line is simply the number of pairs. For each period it should be 5,000 pairs, but occasionally a stock was randomly matched with itself and dropped out. The next line reports how many times the low CAPM beta portfolio had a positive alpha. Followed by the percent of times the low CAPM beta portfolio had a positive alpha. Then we have the average alpha of the low CAPM beta portfolios. Then we have the average CAPM beta of the low CAPM beta portfolios. The last line of the first section is the average market return.

In the next section the number of times various Sharpe ratio orders were recorded. Low-market-high on the left means the low CAPM beta portfolio has the highest Sharpe ratio, followed by the Market portfolio, then the high CAPM beta portfolio. Likewise Low-High-Market means the low CAPM beta portfolio has the highest Sharpe ratio, followed by the high CAPM beta portfolio, followed by the market portfolio, and so on.

Finally, the last section is the percent of time the various Sharpe ratio orders appeared rather than the number of times.

Notice the high number of positive alphas in low CAPM beta portfolios in a rising market, and the one strongly negative alpha occurs in a falling market in the last column on the right. This is a prediction of the model to come.

The volatility effect is also visible in Sharpe ratios, but not as strongly as what we see for alphas. This is a second prediction of the model to come.

By the way, I have done a similar procedure for 60 stock indices with equal weights and random weights where the random weights came from an exponential distribution. The same pattern emerges.

To study the table in detail you will have to stop the video.

Table 10.2B

	Market Return	Average Alpha
1994-1995	1.264%	0.404%
1996-1997	2.136%	0.900%
1998-1999	1.052%	-0.232%
2000-2001	1.404%	0.466%
2002-2003	2.078%	0.658%
2004-2005	1.380%	0.416%
2006-2007	0.714%	0.208%
2008-2009	0.360%	-0.026%
2010-2011	1.072%	0.376%
2012-2013	2.396%	0.798%
2014-2015	0.158%	0.348%
2016-2017	1.632%	0.652%
2018-2019	0.774%	0.360%
Down Market	-3.964%	-0.486%

The graph on the right (below) shows that the magnitude of alpha appears to be linearly associated with the magnitude of the average monthly market returns. The data for the graph is on the left (above).

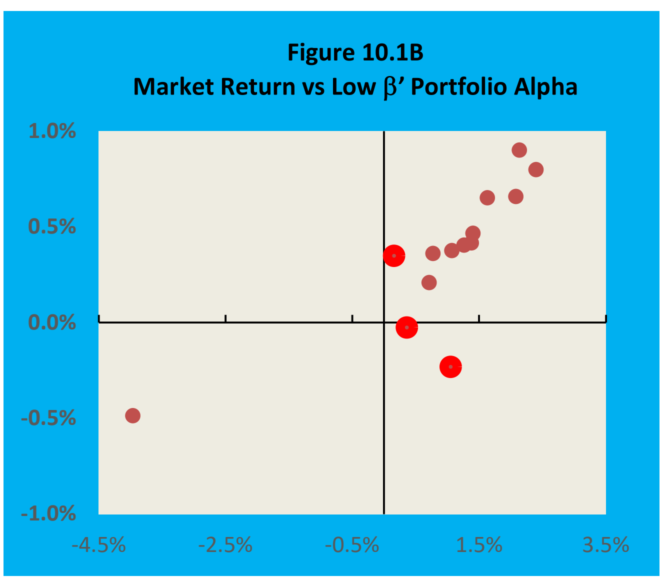


Figure 10.1B
Market Return vs Low β' Portfolio Alpha

Three of the observations do not fit very well with the overall pattern: 1998-1999, 2008-2009, and 2014-2015. The first two of these time periods had unusual price dynamics. If these observations are removed, the remaining observations clearly fall close to a line, as seen on the right.

It will turn out that if the parameters of the model are stable from one period to the next then the expected alpha of one period will be on the same line as the expected alpha of the next time period. It is a third prediction of the model. However, anyone can make a model fit by ignoring inconvenient observations, which means those three observations need to be explained. For the time being I am going to pretend I didn't notice and just get on with it. This issue is revisited in video 27.

In any given two-year period, it is certainly possible that low CAPM beta stocks have a higher alpha than high CAPM beta stocks. There would be nothing interesting in such an observation. But given the method of portfolio construction, and the high frequency of the volatility effect in multiple time periods, it is unlikely that the cause is anything subtle. It must be obvious to be this frequent.

Yet, as we have seen there has been an enormous amount of unsuccessful effort over the past 50 years or so to explain it. It is not obvious.

We need to try something new.

Alpha Found Video 11

The New Model

Welcome once more to Alpha Found.

> "It would be an unsound fancy, and self-contradictory, to expect that things which have never yet been done, can be done, except by means which have never yet been tried."
>
> Francis Bacon, *Novum Organum: Aphorisms Concerning the Interpretation of Nature and the Kingdom of Man*, Aphorism VI, 1620.

Finally, we arrive. The volatility effect has remained mysterious despite the many attempts to explain it. Higher moment models, multi-factor models, and the consequences of unrealistic assumptions, have been examined in the literature for over 50 years to the point of exhaustion. Although these methods have provided numerous insights, at the end of day they don't really capture what we are looking for, which is a clear quantitative relationship.

After looking at the data analysis in the last video, I suggested that the since the effect was obvious, the cause must be obvious also. Yet, it is not obvious.

The new approach, which will be introduced momentarily, I believe will not look promising at first. For that matter, I don't think it will look promising even after this and the next video. However, the harder we push on it, the more promising it will become. We have to go slowly. There will not be much progress in this video.

The key is to separate the market returns into two pieces, such that for a single time period the return is broken up in this manner:

$$R'_{Mt} = R_{Mt} + R_{PMt} \tag{11.1}$$

Or like this as an average over time:

$$\bar{R}'_{M} = \bar{R}_{M} + \bar{R}_{PM} \tag{11.2}$$

The return of the market is easily measured, but what are the two pieces? And what are the subscripts?

I think the best way to do this is to put the two pieces at the heads of two columns and then give the names that have historically been given to each piece.

My apologies for being a little loose with the terminology. Many of the descriptive terms were meant to apply to things other than aggregate equity market returns, and I don't want to try to make others responsible for my thinking, or suggest that their ideas are the same as my ideas, but this should give the flavor of it.

Here we go.

Source	R_M	R_{PM}
Benjamin Graham/Warren Buffett	Fundamental Analysis	Mr. Market
Benjamin Graham/Warren Buffett	Weighing machine	Voting machine
John Maynard Keynes	Economic Expectation	Animal Spirits
Paul Samuelson (Nobel Prize)	Signal	Red Noise
Burton Malkiel	Firm Foundation	Castles in the Air
Alan Greenspan/Robert Shiller	Fundamental Analysis	Irrational Exuberance
Robert Shiller (Nobel Prize)	Fundamentals	Excess Volatility
Richard Roll	Return Generator "G"	

Benjamin Graham and Warren Buffet referred to R_M as fundamental analysis, and R_{PM} as Mr. Market. They also called R_M a weighing machine, and R_{PM} a voting machine. John Maynard Keynes referred to R_M as economic expectation, and R_{PM} as animal spirits. Paul Samuelson called R_M signal, and R_{PM} red noise. Burton Malkiel referred to the pieces as firm foundation vs. castles in the air. Alan Greenspan and Robert Shiller refer to the pieces as fundamental analysis and irrational exuberance. Robert Shiller without Alan Greenspan referred to R_M as fundamentals, and R_{PM} as excess volatility. Richard Roll referred to R_M as the return generator G. He did not have a term for R_{PM} that I know of.

You can see that several prominent figures in the history of investing have commented on the two pieces, or at least commented on something like the two pieces.

As a clarification about G, Richard Roll described G this way: "G is not a portfolio. Instead, G is the unique source of common variation in the ensemble of asset returns. This seemingly innocuous distinction is actually critical." *Ambiguity when Performance is Measured by the Securities Market Line*, Journal of Finance, Volume 33, Issue 4, September 1978, Pgs. 1051-1069.

This list is not complete, nor is it precise. I could add more with the contributions of Nobel Prize winners Daniel Kahneman and Richard Thaler, along with others. The point is simply to give the flavor of the two pieces.

But just to give a few verbal encouragements to the ideas, I have a few quotations for you to consider.

The first is by Irving Fisher who was once described as, "The greatest economist the United States has ever produced." He said,

> "Were it true that each individual speculator made up his mind independently of every other as to the future course of events, the errors of some would probably be offset by the errors of others, but as a matter of fact, the mistakes of the common herd are usually in the same direction. Like sheep they follow single leader."
>
> Irving Fisher, as quoted in *The Myth of the Rational Market*, Justin Fox, 2009, Page 13.

The second quotation comes from Jules Henri Poincare, who has been called, "the Gauss of Modern Mathematics." He also was the thesis advisor of Luis Bachelier. He said,

> "When men are brought together, they no longer decide at random and independently one of another; they influence one another. Multiplex causes come into action. They worry men, dragging them right or left, but one thing there is they cannot destroy, is their Panurge flock of sheep habits. And this is an invariant."
>
> Jules Henri Poincare, *Foundations of Science*, George B. Halsted, Translator, The Science Press, 1913, 1946, Pg. 411

Finally, we have Alan Greenspan, former Chairman of the Federal Reserve for 19 years.

> "But how do we know when irrational exuberance has unduly escalated asset values, which then become subject to unexpected and prolonged contractions as they have in Japan over the past decade? And how do we factor that assessment into monetary policy?"
>
> Alan Greenspan, transcript of a speech given to the American Enterprise Institute, December 5, 1996.

Back to the model.

$$R'_{Mt} = R_{Mt} + R_{PMt} \tag{11.1}$$

$$\bar{R}'_{M} = \bar{R}_{M} + \bar{R}_{PM} \tag{11.2}$$

In some ways to label the pieces is also to libel them, thus prejudging them. So rather than borrow from the list just presented, I will use the following definitions:

R'_{Mt} = the total return of the market in period *t*.

R_{Mt} = the ideal return of the market in period *t*.

R_{PMt} = the perturbation return of the market in period *t*.

The definition of a perturbation is as follows:

1. A disturbance of motion, course, arrangement, or state of equilibrium.
2. **A** disturbance of the regular and usually elliptical course of motion of a celestial body that is produced by some force additional to that which causes its regular motion.

I am going to use the terms *ideal return* and *perturbation return* and correspondingly *ideal systematic risk* and *perturbation risk*.

There could be reasons for perturbation returns that are not behavioral or relating to agency issues, like anticipation of a trend, or economic forecast errors resulting from misjudgment, or unexpected events, or something else no one has thought of. The key is this, perturbation returns have an expected return of zero for the entire market in each time period and across time.

$$E(R_{PMt}) = 0 \tag{11.3}$$

The sum of market perturbations over time approach zero since they are assumed to be governed by a mean reverting process.

$$\sum R_{PMt} \approx 0 \tag{11.4}$$

I am calling the new model the Perturbation Risk Model or PRM.

If $R_{PMt} = 0$ in a given time period, then

$$R'_{Mt} = R_{Mt} + R_{PMt} = R_{Mt} + 0 = R_{Mt}$$

The market return is equal to the ideal return plus the perturbation return, and since the perturbation is zero, all we have is the ideal return, which is equal to the market return.

Therefore, if the perturbations are zero in each time period, then the model reverts back to CAPM and we are in the same place we started. If $R_{PMt} \approx 0$, in each time period then the model is only trivially different from CAPM. If perturbation returns are economically or statistically significant, then the model is economically or statistically significantly different from CAPM.

Since $E(R_{PMt}) = 0$, and the process is mean reverting, then over time the average perturbation return is close to zero, though it is not necessarily or even likely to be zero at any point along the way.

Now is a good time to mention that perturbation risk is not the same as idiosyncratic risk. This is covered in the next video and in several other videos.

If we assume for the moment that the perturbation risk model is correct, then three conclusions follow without further analysis.

1. There are two sources of return for the market in the next time period, ideal return and perturbation return.
2. There is only one source of market return in the long run, ideal return.
3. Ordinary least squares regression will not work as an estimation process.

By the time this series of videos is over, it will be clear that what we have seen so far is sufficient to make the volatility effect a mathematical expectation, but we're a long way from that.

Since I would expect no one to believe this without sufficient evidence, the rest of the videos spell out the details.

Alpha Found Video 12

Perturbation Risk

Welcome once more to part 12.

The last video introduced a new model that allows for returns across the stock market to come from places other than economic fair value returns. If those additional returns are zero in each time period, we're back to CAPM.

Average market returns in the new model have this form, which we saw in the last video.

$$\bar{R}'_M = \bar{R}_M + \bar{R}_{PM} \tag{11.2}$$

Where:

$\bar{R}'_M$ = Average total return of the market

$\bar{R}_M$ = Average ideal total return of the market

$\bar{R}_{PM}$ = Average perturbation return of the market

With this video we begin to put everything into a mathematical framework. There is a lot here and the next video will provide a worked example that will clarify things.

We start with the average ideal return of the market equal to the weighted average of the ideal returns of the constituent securities that make up the market.

The same goes for the average market perturbation return.

$$\bar{R}_M = \sum w_i \, \bar{R}_i \tag{12.1}$$

$$\bar{R}_{PM} = \sum w_i \, \bar{R}_{Pi} \tag{12.2}$$

From the last video, there are two sources of return for the market in the next time period: ideal return and perturbation return.

There is only one source of market return in the long run, ideal return, since the $E(R_{PMt}) = 0$, and the perturbations are mean reverting.

However, there are two sources of variation or risk. This distorts the results from OLS as will be clear. Therefore, ordinary least squares regression will not work as an estimation process.

In this video I continue setting up and defining the pieces of the perturbation risk model.

An ideal market is a market where returns are free from perturbations. There are no ambiguities or unanswered questions about relative performance or relative volatility. Neither is there idiosyncratic risk in an ideal market.

Ideal beta is the sensitivity of a security or portfolio to ideal excess returns. Ideal systematic risk is determined by ideal beta and ideal market risk alone.

Traditional systematic risk is systematic risk as traditionally understood through the lens of CAPM.

Borrowing a phrase from traditional capital market theory, ideal systematic risk is the only risk that gets rewarded over time.

Here is some notation showing the difference between CAPM notation and ideal notation.

CAPM $\beta_L' \; or \; \beta_H'$ Ideal $\beta_L \; or \; \beta_H$

$\sigma_L' \; or \; \sigma_H'$ $\sigma_L \; or \; \sigma_H$

The model developed in the next several videos will be called the forward perturbation risk model. The forward perturbation risk model is an ex-ante model that starts with parameters and works toward expected market statistics.

The backward perturbation risk model is an ex-post model that starts with market statistics and works toward parameter estimates.

If ideal beta is the sensitivity to ideal excess returns, then the returns of the low CAPM beta portfolio using ideal beta will have this form:

$$R_{Lt}' - r_f = \beta_L\left(R_{Mt} - r_f\right) + R_{PLt} \tag{12.3}$$

Where R_{PLt} is the perturbation return for the low CAPM beta portfolio in period *t*.

If we assume that the average perturbation returns are zero over time, ($\bar{R}_{PM} = \bar{R}_{PL} = \bar{R}_{PH} = 0$), then

$$E(\bar{R}_L' | \bar{R}_{PM} = \bar{R}_{PL} = \bar{R}_{PH} = 0) - r_f = \beta_L\left(\bar{R}_M' - r_f\right) \tag{12.4}$$

Because the average ideal return equals the average market return when the average perturbation is zero.

$$\bar{R}_M' = \bar{R}_M + \bar{R}_{PM} = \bar{R}_M + 0 = \bar{R}_M$$

Remember the form of CAPM with alpha:

$$E(\bar{R}_L') - r_f = E(\hat{\alpha}_L') + \beta_L'\left(\bar{R}_M' - r_f\right) \tag{5.3}$$

Which means, if the average perturbations are zero over time ($\bar{R}_{PM} = \bar{R}_{PL} = \bar{R}_{PH} = 0$) the expected return of the low CAPM beta portfolio using the ideal expression can be set equal to the expected return using the CAPM expression, including an alpha term.

$$E(\bar{R}_L') - r_f = \beta_L\left(\bar{R}_M' - r_f\right) = E(\hat{\alpha}_L') + \beta_L'\left(\bar{R}_M' - r_f\right) \tag{12.5a}$$

$$E(\bar{R}_H') - r_f = \beta_H\left(\bar{R}_M' - r_f\right) = E(\hat{\alpha}_H') + \beta_H'\left(\bar{R}_M' - r_f\right) \tag{12.5b}$$

This looks as if in (12.5a) and (12.5b) the original form of CAPM by Sharpe is set equal to the form with alpha. While this appears to be true in this instance, it is not what is going on. If it were, it would tell us nothing about the systematic nature of the volatility effect.

Moving around the terms of (12.5a) and (12.5b), we get this formula for expected alpha:

$$E(\hat{\alpha}_L' | \bar{R}_{PM} = \bar{R}_{PL} = 0) = (\beta_L - \beta_L')\left(\bar{R}_M' - r_f\right) \tag{12.6a}$$

$$E(\hat{\alpha}_H' | \bar{R}_{PM} = \bar{R}_{PH} = 0) = (\beta_H - \beta_H')\left(\bar{R}_M' - r_f\right) \tag{12.6b}$$

Although we are assuming for the time being that the average perturbation over time for both portfolios is zero ($\bar{R}_{PL} = \bar{R}_{PH} = \bar{R}_{PM} = 0$), it's likely that the individual time periods have non-zero perturbations ($R_{PLt} \neq 0,\ R_{PHt} \neq 0$, and $R_{PMt} \neq 0$). This will make all the difference. That is, the aggregate perturbation returns are assumed to be zero over time but not in the individual time periods.

It will be demonstrated in the next couple of videos that alpha has an expectation different from zero, usually.

$$E(\hat{\alpha}'_L) \neq 0$$

We are now entering into the weeds of the model, and will remain there for the rest of the videos. The notation is often subtle.

It's almost certainly true that $\beta'_L \neq \beta_L$ and $\beta'_H \neq \beta_H$ while still maintaining these two identities:

$$\beta'_L + \beta'_H = 2 \qquad \text{from (9.1)}$$

$$\beta_L + \beta_H = 2 \qquad (12.7)$$

In order to get at the quantitative difference between β'_L and β_L, we have to look at their formulas.

The formula for estimated CAPM beta (ex-post) as found in any textbook is as follows:

$$\hat{\beta}'_L = \frac{\hat{\sigma}'_{L,M}}{\hat{\sigma}'^2_M} = \hat{\rho}'_{L,M}\frac{\hat{\sigma}'_L}{\hat{\sigma}'_M} \qquad (6.1)$$

The formula for ideal beta is similar:

$$\beta_L = \frac{\sigma_{L,M}}{\sigma_M^2} = \rho_{L,M}\frac{\sigma_L}{\sigma_M} = \frac{\sigma_L}{\sigma_M} \qquad (12.8)$$

Note the differences.

The formula for CAPM beta is what it has always been.

The formula for ideal beta is in reference to the ideal world.

Here in one line is the derivation of the formula for the covariance between ideal portfolios and the market.

$$\sigma_{L,M} = \sigma_{\beta_L M,M} = \beta_L \sigma_{M,M} = \beta_L \sigma_M^2 \qquad (12.9)$$

Remember that in the ideal returns of the market, there are no error terms or unanswered questions about relative performance.

Which gives us a second derivation for ideal beta:

$$\beta_L = \frac{\sigma_{L,M}}{\sigma_M^2} = \frac{\beta_L \sigma_M^2}{\sigma_M^2} = \frac{\beta_L \sigma_M}{\sigma_M} = \frac{\sigma_L}{\sigma_M}$$

Next is the decomposition of market risk.

$$R'_{Mt} = R_{Mt} + R_{PMt} \qquad (11.1)$$

Take the variance of both sides:

$$\begin{aligned} var(R'_{Mt}) &= var(R_{Mt} + R_{PMt}) \\ &= var(R_{Mt}) + var(R_{PMt}) + 2cov(R_{Mt}, R_{PMt}) \end{aligned}$$

I am assuming that the covariance between the market and the perturbations is zero. Ultimately this is only an okay assumption, but it is good enough for our purposes.

$$var(R'_{Mt}) = \mathrm{var}(R_{Mt}) + var(R_{PMt})$$

Finally, then we switch to a better-looking notation:

$$\sigma'^2_M = \sigma^2_M + \sigma^2_{PM} \tag{12.10}$$

Recall from the video on Jensen's alpha that ex-post return variance of an asset using CAPM components has this derivation:

$$R'_{at} - r_f = \hat{\alpha}'_a + \hat{\beta}'_a\left(R'_{Mt} - r_f\right) + \hat{\varepsilon}'_{at} \tag{5.1}$$

Take the variance of both sides:

$$var\left(R'_a - r_f\right) = var\left(\hat{\alpha}'_a + \hat{\beta}'_a\left(R'_M - r_f\right) + \hat{\varepsilon}'_a\right)$$

Isolated constants drop out:

$$var(R'_a) = var\left(\hat{\beta}'_a(R'_M) + \hat{\varepsilon}'_a\right)$$

Constant coefficients get pulled out and squared:

$$var(R'_a) = \hat{\beta}'^2_a var(R'_M) + var(\hat{\varepsilon}'_a)$$

And of course, we change to a better-looking notation:

$$\hat{\sigma}'^2_a = \hat{\beta}'^2_a \sigma'^2_M + \hat{\sigma}'^2_{a\varepsilon} \tag{5.4}$$

Variance of returns for the low CAPM beta portfolio under the perturbation risk model:

$$R'_{Lt} - r_f = \beta_L\left(R_{Mt} - r_f\right) + R_{PLt} \tag{12.3}$$

Take the variance of both sides:

$$var\left(R'_{Lt} - r_f\right) = var\left(\beta_L\left(R_{Mt} - r_f\right) + R_{PLt}\right)$$

Isolated constants drop out:

$$var(R'_{Lt}) = var(\beta_L(R_{Mt}) + R_{PLt})$$

Constant coefficients get pulled out and squared:

$$var(R'_{Lt}) = \beta^2_L var(R_{Mt}) + var(R_{PLt})$$

And of course, we change to a better-looking notation:

$$\sigma'^2_L = \beta^2_L \sigma^2_M + \sigma^2_{PL} \tag{12.11a}$$

$$\sigma_H'^2 = \beta_H^2 \sigma_M^2 + \sigma_{PH}^2 \tag{12.11b}$$

Under CAPM and PRM the variance equations are structurally identical, and they have the same output, but the inputs are different.

I want to take a few moments to emphasize that perturbation risk is not the same as idiosyncratic risk.

Here are the differences between idiosyncratic risk and perturbation risk.

Idiosyncratic Risk	Perturbation Risk
By construction, residuals sum to zero in each time period	Perturbations likely do not sum to zero in any time period
By construction, idiosyncratic returns sum to zero across a fixed time for each individual security	Perturbations likely do not sum to zero across a selected fixed time for any security, although they could.
Idiosyncratic returns are the same as residuals in OLS	Perturbation returns have nothing to do with OLS except under unlikely and narrow conditions
Idiosyncratic risk is independent from traditional systematic risk	Perturbation risk is a part of traditional systematic risk

By construction, idiosyncratic residuals sum to zero across all securities in each time period. Perturbations likely do not sum to zero in any time period.

By construction, idiosyncratic returns sum to zero across a fixed time for each individual security. Perturbations likely do not sum to zero across a selected fixed time for any security, although they could.

Idiosyncratic returns are another name for residuals in OLS regression. Perturbation returns have nothing to do with OLS, except under unlikely conditions.

Idiosyncratic risk is independent from traditional systematic risk. Perturbation risk is independent from ideal systematic risk, but it is not independent from traditional systematic risk. In fact, it's part of the formula as will be seen in just a moment.

The distinction between perturbation risk and idiosyncratic risk is the key that unlocks the volatility effect and other issues. In a later video I will derive the mathematical relationship between idiosyncratic risk and perturbation risk, but for now it is sufficient to say that they are not at all the same thing and should not be confused.

Back to market perturbation risk.

If an index is divided into two equally weighted pieces, the high CAPM beta portfolio and the low CAPM beta portfolio, then the formula for the variance of the market perturbation is:

$$\sigma_{PM}^2 = \left(\frac{1}{2}\right)^2 \sigma_{PL}^2 + \left(\frac{1}{2}\right)^2 \sigma_{PH}^2 + 2\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\sigma_{PL,PH} = \frac{\sigma_{PL}^2}{4} + \frac{\sigma_{PH}^2}{4} + \frac{\sigma_{PL,PH}}{2}$$

$$\sigma_{PM}^2 = \frac{\sigma_{PL}^2}{4} + \frac{\sigma_{PH}^2}{4} + \frac{\sigma_{PL,PH}}{2} \tag{12.12}$$

For at least several videos I am going to assume that $\sigma_{PL,PH} = 0$.

This is not a good assumption; however, it keeps the calculations and reasoning clean and understandable. This assumption carries the story a long way forward before it needs to be discarded. After I have done what I can with it, and it ultimately fails, it will be much easier to insert the more realistic and complex assumption back in.

So, assuming that $\sigma_{PL,PH} = 0$, we get (12.13):

$$\sigma_{PM}^2 = \frac{\sigma_{PL}^2}{4} + \frac{\sigma_{PH}^2}{4} \tag{12.13}$$

And here is market variance PRM decomposition:

$$\sigma_M'^2 = \sigma_M^2 + \sigma_{PM}^2 = \sigma_M^2 + \frac{\sigma_{PL}^2}{4} + \frac{\sigma_{PH}^2}{4} + \frac{\sigma_{PL,PH}}{2} \tag{12.14}$$

Assuming that $\sigma_{PL,PH} = 0$:

$$\sigma_M'^2 = \sigma_M^2 + \sigma_{PM}^2 = \sigma_M^2 + \frac{\sigma_{PL}^2}{4} + \frac{\sigma_{PH}^2}{4} \tag{12.15}$$

Now for a few items needed to complete the initial set up.

The covariance between the ideal returns of the low CAPM beta portfolio and the high CAPM beta portfolio are as seen here:

$$\sigma_{L,H} = \sigma_{\beta_L M, \beta_H M} = \beta_L \beta_H \sigma_{M,M} = \beta_L \beta_H \sigma_M^2 \tag{12.16}$$

The next one is a little messy so you might want to stop the video at times.

We need the covariance between the low CAPM beta portfolio and the market portfolio.

The covariance between the return of the low CAPM beta portfolio and the market portfolio equals the covariance between the ideal and perturbation returns of the low CAPM beta portfolio and the ideal and perturbation returns of the market.

$$cov(R'_{Lt}, R'_{Mt}) = cov(R_{Lt} + R_{PLt}, (R_{Lt} + R_{PLt} + R_{Ht} + R_{PHt})/2)$$

$$= cov(R_{Lt} + R_{PLt}, (R_{Lt} + R_{PLt})/2) + cov(R_{Lt} + R_{PLt}, (R_{Ht} + R_{PHt})/2)$$

$$= \frac{1}{2}\big(cov(R_{Lt}, R_{Lt}) + cov(R_{Lt}, R_{PLt}) + cov(R_{PLt}, R_{Lt}) + cov(R_{PLt}, R_{PLt}) + cov(R_{Lt}, R_{Ht}) + cov(R_{Lt}, R_{PHt}) + cov(R_{PLt}, R_{Ht}) + cov(R_{PLt}, R_{PHt})\big)$$

Since I am assuming the perturbation returns are independent from the ideal market returns, the parts in red are zero and drop out:

$$= \frac{1}{2}(cov(R_{Lt}, R_{Lt}) + cov(R_{PLt}, R_{PLt}) + cov(R_{Lt}, R_{Ht}) + cov(R_{PLt}, R_{PHt})$$

And with a change in notation,

$$=\frac{1}{2}(\beta_L^2\sigma_M^2 + \sigma_{PL}^2 + \beta_L\beta_H\sigma_M^2 + \sigma_{PL,PH}$$

and a little clean-up it becomes:

$$\sigma'_{L,M} = \beta_L\sigma_M^2 + \frac{1}{2}\sigma_{PL}^2 + \frac{1}{2}\sigma_{PL,PH} \tag{12.17a}$$

$$\sigma'_{H,M} = \beta_H\sigma_M^2 + \frac{1}{2}\sigma_{PH}^2 + \frac{1}{2}\sigma_{PL,PH} \tag{12.17b}$$

Assuming that $\sigma_{PL,PH} = 0$:

$$\sigma'_{L,M} = \beta_L\sigma_M^2 + \frac{1}{2}\sigma_{PL}^2 \tag{12.18a}$$

$$\sigma'_{H,M} = \beta_H\sigma_M^2 + \frac{1}{2}\sigma_{PH}^2 \tag{12.18b}$$

The derivation for the high beta portfolio is identical.

The covariance between the high and low CAPM beta portfolios follows a similar derivation:

$$cov(R'_{Lt}, R'_{Ht}) = cov(R_{Lt} + R_{PLt}, R_{Ht} + R_{PHt})$$

$$= cov(R_{Lt}, R_{Ht}) + cov(R_{Lt}, R_{PHt}) + cov(R_{PLt}, R_{Ht}) + cov(R_{PLt}, R_{PHt})$$

$$\sigma'_{L,H} = \beta_L\beta_H\sigma_M^2 + \sigma_{PL,PH} \tag{12.19}$$

Assuming that $\sigma_{PL,PH} = 0$:

$$\sigma'_{L,H} = \beta_L\beta_H\sigma_M^2 \tag{12.20}$$

Which is the same as the formula for ideal covariance between the two portfolios.

Finally, using this information, I want to show the formula for CAPM beta written in the mathematical vocabulary of the perturbation risk model:

$$\beta'_L = \frac{\sigma'_{L,M}}{\sigma'^2_M} = \frac{\beta_L\sigma_M^2 + \frac{1}{2}\sigma_{PL}^2 + \frac{1}{2}\sigma_{PL,PH}}{\sigma'^2_M} \tag{12.21a}$$

$$\beta'_H = \frac{\sigma'_{H,M}}{\sigma'^2_M} = \frac{\beta_H\sigma_M^2 + \frac{1}{2}\sigma_{PH}^2 + \frac{1}{2}\sigma_{PL,PH}}{\sigma'^2_M} \tag{12.21b}$$

Assuming that $\sigma_{PL,PH} = 0$:

$$\beta'_L = \frac{\beta_L\sigma_M^2 + \frac{1}{2}\sigma_{PL}^2}{\sigma'^2_M} \tag{12.22a}$$

$$\beta'_H = \frac{\beta_H\sigma_M^2 + \frac{1}{2}\sigma_{PH}^2}{\sigma'^2_M} \tag{12.22b}$$

Stated without derivation:

$$\rho'_{L,M} = \frac{\beta_L \sigma_M^2 + \frac{1}{2}\sigma_{PL}^2 + \frac{1}{2}\sigma_{PL,PH}}{\sigma'_L \sigma'_M} \tag{12.23}$$

Assuming that $\sigma_{PL,PH} = 0$:

$$\rho'_{L,M} = \frac{\beta_L \sigma_M^2 + \frac{1}{2}\sigma_{PL}^2}{\sigma'_L \sigma'_M} \tag{12.24a}$$

$$\rho'_{H,M} = \frac{\beta_H \sigma_M^2 + \frac{1}{2}\sigma_{PH}^2}{\sigma'_H \sigma'_M} \tag{12.24b}$$

In this video there have been a whole lot of formulas without calculations.

The next video will have an example that will clarify what it all means, but it will be only a piece of the story.

Before I end, here is a summary of the formulas set to music.

Average market return decomposition	$\bar{R}'_M = \bar{R}_M + \bar{R}_{PM}$	(11.2)
PRM return decomposition	$R'_{Lt} - r_f = \beta_L(\bar{R}_M - r_f) + R_{PLt}$	(12.3)
Conditional PRM return decomposition	$E(\bar{R}'_L \mid \bar{R}_{PM} = \bar{R}_{PL} = 0) - r_f = \beta_L(\bar{R}'_M - r_f)$	(12.4)
Conditional expected CAPM alpha	$E(\hat{\alpha}'_L \mid \bar{R}_{PM} = \bar{R}_{PL} = 0) = (\beta_L - \beta'_L)(\bar{R}'_M - r_f)$	(12.6a)
Ideal beta	$\beta_L = \frac{\sigma_{L,M}}{\sigma_M^2} = \rho_{L,M}\frac{\sigma_L}{\sigma_M} = \frac{\sigma_L}{\sigma_M}$	(12.8)
Covariance between the ideal returns of the low CAPM beta portfolio and market portfolios	$\sigma_{L,M} = \beta_L \sigma_M^2$	(12.9)
Covariance between the ideal returns of the low and high CAPM beta portfolios	$\sigma_{L,H} = \beta_L \beta_H \sigma_M^2$	(12.16)
PRM market risk decomposition (1st form)	$\sigma_M'^2 = \sigma_M^2 + \sigma_{PM}^2$	(12.10)
Portfolio risk decomposed with CAPM components	$\sigma_L'^2 = \beta_L'^2 \sigma_M'^2 + \sigma_{L\varepsilon}'^2$	(5.4)
Portfolio risk decomposed with PRM Components	$\sigma_L'^2 = \beta_L^2 \sigma_M^2 + \sigma_{PL}^2$	(12.11a)
Market perturbation risk decomposition	$\sigma_{PM}^2 = \frac{\sigma_{PL}^2}{4} + \frac{\sigma_{PH}^2}{4} + \frac{\sigma_{PL,PH}}{2}$	(12.12)
PRM market risk decomposition (2nd form)	$\sigma_M'^2 = \sigma_M^2 + \sigma_{PM}^2 = \sigma_M^2 + \frac{\sigma_{PL}^2}{4} + \frac{\sigma_{PH}^2}{4} + \frac{\sigma_{PL,PH}}{2}$	(12.14)
Assuming $\sigma_{PL,PH} = 0$, then	$\sigma_M'^2 = \sigma_M^2 + \sigma_{PM}^2 = \sigma_M^2 + \frac{\sigma_{PL}^2}{4} + \frac{\sigma_{PH}^2}{4}$	(12.15)

Covariance between the low CAPM beta portfolio and the market	$\sigma'_{L,M} = \beta_L \sigma_M^2 + \frac{1}{2}\sigma_{PL}^2 + \frac{1}{2}\sigma_{PL,PH}$	(12.17a)
Assuming $\sigma_{PL,PH} = 0$, then	$\sigma'_{L,M} = \beta_L \sigma_M^2 + \frac{1}{2}\sigma_{PL}^2$	(12.18a)
	$\sigma'_{H,M} = \beta_H \sigma_M^2 + \frac{1}{2}\sigma_{PH}^2$	(12.18b)
Covariance between the low and high CAPM beta portfolios	$\sigma'_{L,H} = \beta_L \beta_H \sigma_M^2 + \sigma_{PL,PH}$	(12.19)
Assuming that $\sigma_{PL,PH} = 0$, then	$\sigma'_{L,H} = \sigma_L \sigma_H = \beta_L \beta_H \sigma_M^2$	(12.20)
CAPM beta written with PRM components	$\beta'_L = \frac{\sigma'_{L,M}}{\sigma'^2_M} = \frac{\beta_L \sigma_M^2 + \frac{1}{2}\sigma_{PL}^2 + \frac{1}{2}\sigma_{PL,PH}}{\sigma'^2_M}$	(12.21a)
Assuming that $\sigma_{PL,PH} = 0$, then	$\beta'_L = \frac{\sigma'_{L,M}}{\sigma'^2_M} = \frac{\beta_L \sigma_M^2 + \frac{1}{2}\sigma_{PL}^2}{\sigma'^2_M}$	(12.22a)
Assuming that $\sigma_{PL,PH} = 0$, then	$\beta'_H = \frac{\sigma'_{H,M}}{\sigma'^2_M} = \frac{\beta_H \sigma_M^2 + \frac{1}{2}\sigma_{PH}^2}{\sigma'^2_M}$	(12.22b)
Correlation between the low CAPM beta portfolio and the market	$\rho'_{L,M} = \frac{\beta_L \sigma_M^2 + \frac{1}{2}\sigma_{PL}^2 + \frac{1}{2}\sigma_{PL,PH}}{\sigma'_L \sigma'_M}$	(12.23)
Assuming $\sigma_{PL,PH} = 0$, then	$\rho'_{L,M} = \frac{\beta_L \sigma_M^2 + \frac{1}{2}\sigma_{PL}^2}{\sigma'_L \sigma'_M}$	(12.24a)

Alpha Found Video 13

An Example

Welcome once more to Alpha Found part 13.

I want to begin by putting up the formulas from the last video.

Average market return decomposition	$\bar{R}'_M = \bar{R}_M + \bar{R}_{PM}$	(11.2)
PRM return decomposition	$R'_{Lt} - r_f = \beta_L(\bar{R}_M - r_f) + R_{PLt}$	(12.3)
Conditional PRM return decomposition	$E(\bar{R}'_L \mid \bar{R}_{PM} = \bar{R}_{PL} = 0) - r_f = \beta_L(\bar{R}'_M - r_f)$	(12.4)
Conditional expected CAPM alpha	$E(\hat{\alpha}'_L \mid \bar{R}_{PM} = \bar{R}_{PL} = 0) = (\beta_L - \beta'_L)(\bar{R}'_M - r_f)$	(12.6a)
Ideal beta	$\beta_L = \frac{\sigma_{L,M}}{\sigma_M^2} = \rho_{L,M}\frac{\sigma_L}{\sigma_M} = \frac{\sigma_L}{\sigma_M}$	(12.8)
Covariance between the ideal returns of the low CAPM beta portfolio and market portfolios	$\sigma_{L,M} = \beta_L \sigma_M^2$	(12.9)
Covariance between the ideal returns of the low and high CAPM beta portfolios	$\sigma_{L,H} = \beta_L \beta_H \sigma_M^2$	(12.16)
PRM market risk decomposition (1st form)	$\sigma_M'^2 = \sigma_M^2 + \sigma_{PM}^2$	(12.10)
Portfolio risk decomposed with CAPM components	$\sigma_L'^2 = \beta_L'^2 \sigma_M'^2 + \sigma_{L\varepsilon}'^2$	(5.4)
Portfolio risk decomposed with PRM (12.11a) Components	$\sigma_L'^2 = \beta_L^2 \sigma_M^2 + \sigma_{PL}^2$	
Market perturbation risk decomposition	$\sigma_{PM}^2 = \frac{\sigma_{PL}^2}{4} + \frac{\sigma_{PH}^2}{4} + \frac{\sigma_{PL,PH}}{2}$	(12.12)
PRM market risk decomposition (2nd form)	$\sigma_M'^2 = \sigma_M^2 + \sigma_{PM}^2 = \sigma_M^2 + \frac{\sigma_{PL}^2}{4} + \frac{\sigma_{PH}^2}{4} + \frac{\sigma_{PL,PH}}{2}$	(12.14)
Assuming $\sigma_{PL,PH} = 0$, then	$\sigma_M'^2 = \sigma_M^2 + \sigma_{PM}^2 = \sigma_M^2 + \frac{\sigma_{PL}^2}{4} + \frac{\sigma_{PH}^2}{4}$	(12.15)
Covariance between the low CAPM beta portfolio and the market	$\sigma'_{L,M} = \beta_L \sigma_M^2 + \frac{1}{2}\sigma_{PL}^2 + \frac{1}{2}\sigma_{PL,PH}$	(12.17a)
Assuming $\sigma_{PL,PH} = 0$, then	$\sigma'_{L,M} = \beta_L \sigma_M^2 + \frac{1}{2}\sigma_{PL}^2$	(12.18a)
	$\sigma'_{H,M} = \beta_H \sigma_M^2 + \frac{1}{2}\sigma_{PH}^2$	(12.18b)
Covariance between the low and high CAPM beta portfolios	$\sigma'_{L,H} = \beta_L \beta_H \sigma_M^2 + \sigma_{PL,PH}$	(12.19)

Assuming that $\sigma_{PL,PH} = 0$, then $$\sigma'_{L,H} = \sigma_L \sigma_H = \beta_L \beta_H \sigma_M^2 \quad (12.20)$$

CAPM beta written with PRM components $$\beta'_L = \frac{\sigma'_{L,M}}{\sigma'^2_M} = \frac{\beta_L \sigma_M^2 + \frac{1}{2}\sigma_{PL}^2 + \frac{1}{2}\sigma_{PL,PH}}{\sigma'^2_M} \quad (12.21a)$$

Assuming that $\sigma_{PL,PH} = 0$, then $\beta'_L = \frac{\sigma'_{L,M}}{\sigma'^2_M} = \frac{\beta_L \sigma_M^2 + \frac{1}{2}\sigma_{PL}^2}{\sigma'^2_M}$ (12.22a)

Assuming that $\sigma_{PL,PH} = 0$, then $\beta'_H = \frac{\sigma'_{H,M}}{\sigma'^2_M} = \frac{\beta_H \sigma_M^2 + \frac{1}{2}\sigma_{PH}^2}{\sigma'^2_M}$ (12.22b)

Correlation between the low CAPM beta portfolio and the market $$\rho'_{L,M} = \frac{\beta_L \sigma_M^2 + \frac{1}{2}\sigma_{PL}^2 + \frac{1}{2}\sigma_{PL,PH}}{\sigma'_L \sigma'_M} \quad (12.23)$$

Assuming $\sigma_{PL,PH} = 0$, then $$\rho'_{L,M} = \frac{\beta_L \sigma_M^2 + \frac{1}{2}\sigma_{PL}^2}{\sigma'_L \sigma'_M} \quad (12.24a)$$

Now we need an example. It makes several points more clearly than can be made by algebraic manipulation alone. As always, I recommend you stop the video to make sure you understand the calculations.

The example will show how the forward perturbation risk model works, how it is different from CAPM, how CAPM beta does not equal ideal beta ($\beta'_L \neq \beta_L$), and how the expected alpha is usually different from zero ($E(\hat{\alpha}') \neq 0$) for each security or portfolio.

Example 13.1

Assume the following ideal parameters values:

$\beta_L = 0.5$ $\quad$ $\beta_H = 1.5$

$\bar{R}'_M - r_f = 10\%$ $\quad$ $\bar{R}_{PM} = 0\%$

$\sigma_M = 18\%$ $\quad$ $\sigma_L = \beta_L \sigma_M = 9\%$ $\quad$ $\sigma_H = \beta_H \sigma_M = 27\%$

$\sigma_{PL} = 3.33\%$ $\quad$ $\sigma_{PH} = 16.66\%$

And, although some will find it annoying, I'll be using far too many digits past the decimal point.

With those inputs, the following calculations are made using the formulas derived so far:

$$\sigma'_M = \sqrt{\sigma_M^2 + \frac{\sigma_{PL}^2 + \sigma_{PH}^2}{4}} = 19.90528\% \qquad \sigma'_{L,H} = \sigma_L \sigma_H = 2.4300\%$$

$$\sigma'_L = \sqrt{\sigma_L^2 + \sigma_{PL}^2} = 9.59745\% \qquad \sigma'_H = \sqrt{\sigma_H^2 + \sigma_{PH}^2} = 31.72976\%$$

$$\sigma'_{L,M} = \beta_L \sigma_M^2 + \frac{1}{2}\sigma_{PL}^2 = 1.6755\% \qquad \sigma'_{H,M} = \beta_H \sigma_M^2 + \frac{1}{2}\sigma_{PH}^2 = 6.2488\%$$

$$\beta'_L = \frac{\sigma'_{L,M}}{\sigma'^2_M} = \frac{\beta_L\sigma^2_M + \frac{1}{2}\sigma^2_{PL}}{\sigma'^2_M} = 0.42288 \qquad \beta'_H = \frac{\sigma'_{H,M}}{\sigma'^2_M} = \frac{\beta_H\sigma^2_M + \frac{1}{2}\sigma^2_{PH}}{\sigma'^2_M} = 1.57712$$

$$\bar{R}_{PM} = 0 \qquad E(\hat{\alpha}'_L) = (\beta_L - \beta'_L)\left(\bar{R}'_M - r_f\right) = 0.7712\%$$

Here is the summary table:

$$\sigma'_M = \sqrt{\sigma^2_M + \frac{\sigma^2_{PL} + \sigma^2_{PH}}{4}} = 19.90528\% \qquad \sigma'_{L,H} = \sigma_L\sigma_H = 2.4300\%$$

$$\sigma'_L = \sqrt{\sigma^2_{PL} + \sigma^2_{PL}} = 9.59745\% \qquad \sigma'_H = \sqrt{\sigma^2_H + \sigma^2_{PH}} = 31.72976\%$$

$$\sigma'_{L,M} = \beta_L\sigma^2_M + \frac{1}{2}\sigma^2_{PL} = 1.6755\% \qquad \sigma'_{H,M} = \beta_H\sigma^2_M + \frac{1}{2}\sigma^2_{PH} = 6.2488\%$$

$$\beta'_L = \frac{\sigma'_{L,M}}{\sigma'^2_M} = \frac{\beta_L\sigma^2_M + \frac{1}{2}\sigma^2_{PL}}{\sigma'^2_M} = 0.42288 \qquad \beta'_H = \frac{\sigma'_{H,M}}{\sigma'^2_M} = \frac{\beta_H\sigma^2_M + \frac{1}{2}\sigma^2_{PH}}{\sigma'^2_M} = 1.57712$$

$$E(\hat{\alpha}'_L) = (\beta_L - \beta'_L)\left(\bar{R}'_M - r_f\right) = 0.7712\% \qquad E(\hat{\alpha}'_H) = (\beta_H - \beta'_H)\left(\bar{R}'_M - r_f\right) = -0.7712\%$$

The expected return of the low CAPM beta portfolio minus the risk-free rate using the perturbation risk model equals:

$$E(\bar{R}'_L) - r_f = \beta_L\left(\bar{R}'_M - r_f\right) = .5(10\%) = 5\%$$

Simultaneously, using CAPM, the expected return is:

$$E(\bar{R}'_L) - r_f = E(\alpha'_L) + \beta'_L\left(\bar{R}'_M - r_f\right) = 0.7712\% + .42288(10\%) = 5\%$$

Under these conditions, CAPM has $E(\hat{\alpha}'_L)$ = 0.7712%, which would ordinarily be thought of as excess risk-adjusted return. The lens of perturbation risk shows that $E(\hat{\alpha}'_L)$ is an expectation without there being any actual excess risk-adjusted return. The expected return under both models is the same, but CAPM has an expected alpha (interpreted as excess risk-adjusted return) while PRM does not have excess risk-adjusted return (at least in this example).

The same calculation for the high β' portfolio results in $E(\hat{\alpha}'_H) = -0.7712\%$.

Therefore:

$$E(\hat{\alpha}'_L) > 0 > E(\hat{\alpha}'_H) \tag{13.1}$$

The expected ideal Sharpe ratios are as follows:

$$E(\hat{S}_L) = E(\hat{S}_M) = E(\hat{S}_H) \quad \rightarrow \quad \frac{5\%}{9\%} = \frac{10\%}{18\%} = \frac{15\%}{27\%} = 55.56\%$$

Notice that they're equal.

However, the inclusion of perturbation risk gives a different expectation:

$$E(\hat{S}'_L) = \frac{\bar{R}'_L - r_f}{\sigma'_L} = \frac{5\%}{9.597\%} = 52.10\%,$$

$$E(\hat{S}'_H) = \frac{\bar{R}'_H - r_f}{\sigma'_H} = \frac{15\%}{31.730\%} = 47.28\%$$

$$E(\hat{S}'_M) = \frac{\bar{R}'_M - r_f}{\sigma'_M} = \frac{10\%}{19.905\%} = 50.24\%$$

Therefore, in the marketplace, under the conditions given, without any excess return after adjusting for ideal systematic risk, the Sharpe ratios have the following expected relationship:

$$E(\hat{S}'_L) > E(\hat{S}'_M) > E(\hat{S}'_H) \qquad (13.2)$$

Under the conditions given, the PRM predicts as a mathematical expectation, the volatility effect.

$$E(\hat{\alpha}'_L) > 0 > E(\hat{\alpha}'_H) \qquad (13.1)$$

$$E(\hat{S}'_L) > E(\hat{S}'_M) > E(\hat{S}'_H) \qquad (13.2)$$

This example demonstrates the mechanics of the perturbation risk model. It also shows how OLS regression as an estimator for CAPM parameters generates non-zero alphas as an expectation, though there are no excess return.

Alpha as an economic reality has been shown in this example to be a statistical artifact of OLS, which in this context is misapplied.

Much more detail is needed. Much more is coming.

Alpha Found Video 14

More Detail Part 1

Welcome once more to part 14 of Alpha Found.

The example in the last video demonstrates the mechanics of the perturbation risk model. It also demonstrates how a positive alpha does not necessarily mean excess risk adjusted return.

If the perturbation risk model is correct, one consequence is that the expected non-zero alpha is simply a statistical artifact of using OLS as an estimation method.

Now we need to go from the numerical example in the last video back to algebraic reasoning. This will take some time. It is not possible simply to watch the video and understand it, the mathematics moves to fast. I recommend you go to the associated website (www.RealAlphaFound.com) and download the transcript so you can study the derivations more carefully. There is no login and you will not be asked for any personal information.

In this video we will begin to derive various expected outcomes depending on the magnitude and relationship between the perturbation risks.

If an index is divided into two portfolios of equal market capitalization, there are nine scenarios or combinations of expected outcomes (assuming $\bar{R}_{PM} = \bar{R}_{PL} = \bar{R}_{PH} = 0$) that can occur simply by having the requisite ideal systematic risk and perturbation risk parameters without altering average arithmetic returns.

Let's take a look.

The first four scenarios are contrary to the volatility effect:

1. $$E(\hat{\alpha}'_L) < 0 < E(\hat{\alpha}'_H) \quad \text{and} \quad E(\hat{S}'_H) > \boldsymbol{E(\hat{S}'_M)} > E(\hat{S}'_L) \tag{14.1}$$

2. $$E(\hat{\alpha}'_L) < 0 < E(\hat{\alpha}'_H) \quad \text{and} \quad E(\hat{S}'_H) = \boldsymbol{E(\hat{S}'_M)} > E(\hat{S}'_L) \tag{14.2}$$

3. $$E(\hat{\alpha}'_L) < 0 < E(\hat{\alpha}'_H) \quad \text{and} \quad \boldsymbol{E(\hat{S}'_M)} > E(\hat{S}'_H) > E(\hat{S}'_L) \tag{14.3}$$

4. $$E(\hat{\alpha}'_L) = 0 = E(\hat{\alpha}'_H) \quad \text{and} \quad \boldsymbol{E(\hat{S}'_M)} > E(\hat{S}'_H) > E(\hat{S}'_L) \tag{14.4}$$

In scenarios 1 – 3, $E(\hat{\alpha}'_L) < 0 < E(\hat{\alpha}'_H)$, and $E(\hat{S}'_L)$ is the smallest of the Sharpe ratios. Take a moment to notice how the order and relationship between Sharpe ratios change with each scenario.

The fourth scenario is CAPM with the expected alphas equal to zero.

5. $$E(\hat{\alpha}'_L) > 0 > E(\hat{\alpha}'_H) \quad \text{and} \quad \boldsymbol{E(\hat{S}'_M)} > E(\hat{S}'_H) > E(\hat{S}'_L) \tag{14.5}$$

6. $$E(\hat{\alpha}'_L) > 0 > E(\hat{\alpha}'_H) \quad \text{and} \quad \boldsymbol{E(\hat{S}'_M)} > E(\hat{S}'_H) = E(\hat{S}'_L) \tag{14.6}$$

The fifth and sixth scenarios are weakly consistent with the volatility effect. $E(\hat{\alpha}'_L) > 0$, but $E(\hat{S}'_L)$ is still the smallest. Note that scenario 6 is the same as line (9.17).

7. $E(\hat{\alpha}_L') > 0 > E(\hat{\alpha}_H')$ and $\boldsymbol{E(\widehat{S}_M')} > E(\hat{S}_L') > E(\hat{S}_H')$ (14.7)

8. $E(\hat{\alpha}_L') > 0 > E(\hat{\alpha}_H')$ and $\boldsymbol{E(\widehat{S}_M')} = E(\hat{S}_L') > E(\hat{S}_H')$ (14.8)

9. $E(\hat{\alpha}_L') > 0 > E(\hat{\alpha}_H')$ and $E(\hat{S}_L') > \boldsymbol{E(\widehat{S}_M')} > E(\hat{S}_H')$ (14.9)

The remaining three scenarios, which describe the volatility effect as commonly understood, are what is usually found in market data. Scenario 9 is the one that will eventually have our attention, where the $E(\hat{S}_L')$ is highest.

It is the average excess market return $(\bar{R}_M' - r_f)$ (assumed to be greater than 0), the ratio of the two perturbation risks σ_{PH}/σ_{PL}, called the *perturbation risk ratio* or *PRR*, and their relationship to the two ideal βs, that determine the relationships between expected alphas on the one hand, and expected Sharpe ratios on the other.

There are several ways the following conditions and derivations could be presented.

I have chosen to do it in a way that I believe is the best suited for these videos, but that might not be the best way for other purposes. In a later video it will be clear why I have chosen the way I have.

Remember that we are still assuming that the average perturbation over time is zero, but not zero in any given time period.

Scenario 1

Let's derive the condition of Scenario 1. After the derivation is finished, it might take some review to understand it fully.

Scenario 1 has $E(\hat{\alpha}_L') < 0$, which is less than $E(\hat{\alpha}_H')$. And the $E(\hat{S}_H')$ is greater than $E(\hat{S}_M')$, which is greater than $E(\hat{S}_L')$.

$$E(\hat{\alpha}_L') < 0 < E(\hat{\alpha}_H') \quad \text{and} \quad E(\hat{S}_H') > E(\hat{S}_M') > E(\hat{S}_L') \tag{14.1}$$

If $E(\hat{S}_H') > E(\hat{S}_M')$, we can use the definition of the Sharpe ratio to get this expression:

$$\frac{\alpha_H' + \beta_H'(\bar{R}_M' - r_f)}{\sigma_H'} > \frac{(\bar{R}_M' - r_f)}{\sigma_M'}$$

Using equation (12.5b) we can make a substitution:

$$\alpha_H' + \beta_H'(\bar{R}_M' - r_f) = \beta_H(\bar{R}_M' - r_f) \quad \rightarrow \quad \frac{\beta_H(\bar{R}_M' - r_f)}{\sigma_H'} > \frac{(\bar{R}_M' - r_f)}{\sigma_M'}$$

Then by dividing through by average excess market return we get:

$$\frac{\beta_H}{\sigma_H'} > \frac{1}{\sigma_M'}$$

Squaring both sides, taking the reciprocal, and reversing the inequality gives:

$$\frac{\beta_H^2}{\sigma_H'^2} > \frac{1}{\sigma_M'^2} \quad \rightarrow \quad \frac{\sigma_H'^2}{\beta_H^2} < \sigma_M'^2$$

Then by expanding the variance on both sides, and canceling β_H^2 on the left-hand side:

$$\frac{\beta_H^2\sigma_M^2+\sigma_{PH}^2}{\beta_H^2} < \sigma_M^2 + \frac{\sigma_{PL}^2}{4} + \frac{\sigma_{PH}^2}{4} + \frac{\sigma_{PL,PH}}{2} \quad \rightarrow \quad \sigma_M^2 + \frac{\sigma_{PH}^2}{\beta_H^2} < \sigma_M^2 + \frac{\sigma_{PL}^2}{4} + \frac{\sigma_{PH}^2}{4} + \frac{\sigma_{PL,PH}}{2}$$

Subtracting ideal market variance from both sides:

$$\frac{\sigma_{PH}^2}{\beta_H^2} < \frac{\sigma_{PL}^2}{4} + \frac{\sigma_{PH}^2}{4} + \frac{\sigma_{PL,PH}}{2}$$

And multiplying through by 4:

$$\frac{4\sigma_{PH}^2}{\beta_H^2} < \sigma_{PL}^2 + \sigma_{PH}^2 + 2\sigma_{PL,PH}$$

Dividing through by perturbation risk of the high CAPM beta portfolio (σ_{PH}^2) and subtracting 1 from both sides gives:

$$\frac{4}{\beta_H^2} < \frac{\sigma_{PL}^2+2\sigma_{PL,PH}}{\sigma_{PH}^2} + 1 \quad \rightarrow \quad \frac{4}{\beta_H^2} - 1 < \frac{\sigma_{PL}^2+2\sigma_{PL,PH}}{\sigma_{PH}^2}$$

$$\frac{4-\beta_H^2}{\beta_H^2} < \frac{\sigma_{PL}^2+2\sigma_{PL,PH}}{\sigma_{PH}^2}$$

Finally, taking the reciprocal again and reversing the inequality:

$$\frac{\beta_H^2}{4-\beta_H^2} > \frac{\sigma_{PH}^2}{\sigma_{PL}^2+2\sigma_{PL,PH}}$$

I want to reorganize the left-hand side for reasons that are not obvious:

$$\frac{\beta_H^2}{4-\beta_H^2} = \frac{\beta_H^2}{(2-\beta_H)(2+\beta_H)} = \frac{\beta_H^2}{(\beta_L)(\beta_L+\beta_H+\beta_H)} = \frac{\beta_H^2}{\beta_L^2+2\beta_L\beta_H}$$

Substituting the right-hand term back into the previous inequality, gives us this form:

$$\frac{\beta_H^2}{\beta_L^2+2\beta_L\beta_H} > \frac{\sigma_{PH}^2}{\sigma_{PL}^2+2\sigma_{PL,PH}}$$

Taking the square root of both sides:

$$\frac{\beta_H}{\sqrt{\beta_L^2+2\beta_L\beta_H}} > \frac{\sigma_{PH}}{\sqrt{\sigma_{PL}^2+2\sigma_{PL,PH}}} \tag{14.10}$$

Assuming $\sigma_{PL,PH} = 0$:

$$\frac{\beta_H}{\sqrt{\beta_L^2+2\beta_L\beta_H}} > \frac{\sigma_{PH}}{\sigma_{PL}} \tag{14.11}$$

Using the ideal betas from example 13.1 ($\beta_L = 0.5$ and $\beta_H = 1.5$), the left-hand condition is:

$$\frac{\beta_H}{\sqrt{\beta_L^2 + 2\beta_L\beta_H}} = \frac{1.5}{\sqrt{.5^2 + 2(.5)(1.5)}} = 1.1339 > \frac{\sigma_{PH}}{\sigma_{PL}}$$

As a numerical example, if $\sigma_{PH} = 10\%$ and $\sigma_{PL} = 10\%$, then this condition is met:

$$1.1339 > \frac{\sigma_{PH}}{\sigma_{PL}} = \frac{10\%}{10\%} = 1.00$$

If condition 1 is met, then scenario 1 is expected.

If we take the numbers from this quick example and plug them into the formulas of videos 12 and 13, we get:

$$E(\hat{\alpha}_L') = -0.668\% \qquad E(\hat{\alpha}_H') = 0.668\%$$

$$E(\hat{S}_H') = 0.5210 \qquad E(\hat{S}_M') = 0.5171 \qquad E(\hat{S}_L') = 0.3716$$

Which means:

$$E(\hat{\alpha}_L') < 0 < E(\hat{\alpha}_H') \quad \text{and} \quad E(\hat{S}_H') > E(\hat{S}_M') > E(\hat{S}_L')$$

For the same average arithmetic total returns and ideal risks as found in example 13.1, expected alphas and expected Sharpe ratios can change in magnitude and order depending on the relationship between the perturbation risks.

Alpha Found Video 15

More Detail Part 2

Welcome again.

In the last video I listed the 9 scenarios for expected alpha and Sharpe ratios. The average arithmetic return is unchanging in each scenario, but the perturbation risk ratio is different in each scenario.

Condition 1 was derived in the last video. In this video conditions 2–5, which will lead to scenarios 2-5, will be derived.

As a reminder, here are the 9 scenarios (assuming $\bar{R}_{PM} = \bar{R}_{PL} = \bar{R}_{PH} = 0$) which define the relationships between possible expected outcomes.

Four scenarios contrary to the volatility effect:

1. $$E(\hat{\alpha}'_L) < 0 < E(\hat{\alpha}'_H) \quad \text{and} \quad E(\hat{S}'_H) > \boldsymbol{E}(\widehat{\boldsymbol{S}}'_{\boldsymbol{M}}) > E(\hat{S}'_L) \tag{14.1}$$

2. $$E(\hat{\alpha}'_L) < 0 < E(\hat{\alpha}'_H) \quad \text{and} \quad E(\hat{S}'_H) = \boldsymbol{E}(\widehat{\boldsymbol{S}}'_{\boldsymbol{M}}) > E(\hat{S}'_L) \tag{14.2}$$

3. $$E(\hat{\alpha}'_L) < 0 < E(\hat{\alpha}'_H) \quad \text{and} \quad \boldsymbol{E}(\widehat{\boldsymbol{S}}'_{\boldsymbol{M}}) > E(\hat{S}'_H) > E(\hat{S}'_L) \tag{14.3}$$

4. $$E(\hat{\alpha}'_L) = 0 = E(\hat{\alpha}'_H) \quad \text{and} \quad \boldsymbol{E}(\widehat{\boldsymbol{S}}'_{\boldsymbol{M}}) > E(\hat{S}'_H) > E(\hat{S}'_L) \tag{14.4}$$

Two scenarios weakly consistent with the volatility effect:

5. $$E(\hat{\alpha}'_L) > 0 > E(\hat{\alpha}'_H) \quad \text{and} \quad \boldsymbol{E}(\widehat{\boldsymbol{S}}'_{\boldsymbol{M}}) > E(\hat{S}'_H) > E(\hat{S}'_L) \tag{14.5}$$

6. $$E(\hat{\alpha}'_L) > 0 > E(\hat{\alpha}'_H) \quad \text{and} \quad \boldsymbol{E}(\widehat{\boldsymbol{S}}'_{\boldsymbol{M}}) > E(\hat{S}'_H) = E(\hat{S}'_L) \tag{14.6}$$

Three scenarios consistent with the volatility effect:

7. $$E(\hat{\alpha}'_L) > 0 > E(\hat{\alpha}'_H) \quad \text{and} \quad \boldsymbol{E}(\widehat{\boldsymbol{S}}'_{\boldsymbol{M}}) > E(\hat{S}'_L) > E(\hat{S}'_H) \tag{14.7}$$

8. $$E(\hat{\alpha}'_L) > 0 > E(\hat{\alpha}'_H) \quad \text{and} \quad \boldsymbol{E}(\widehat{\boldsymbol{S}}'_{\boldsymbol{M}}) = E(\hat{S}'_L) > E(\hat{S}'_H) \tag{14.8}$$

9. $$E(\hat{\alpha}'_L) > 0 > E(\hat{\alpha}'_H) \quad \text{and} \quad E(\hat{S}'_L) > \boldsymbol{E}(\widehat{\boldsymbol{S}}'_{\boldsymbol{M}}) > E(\hat{S}'_H) \tag{14.9}$$

Scenario 2

Let's look at scenario 2:

$$E(\hat{\alpha}'_L) < 0 < E(\hat{\alpha}'_H) \quad \text{and} \quad E(\hat{S}'_H) = E(\hat{S}'_M) > E(\hat{S}'_L) \tag{14.2}$$

Conveniently, deriving the conditions of scenario 2 is identical to the derivation for the conditions of scenario 1, with an equal sign replacing the inequality at every step, resulting in this equation:

$$\frac{\beta_H}{\sqrt{\beta_L^2+2\beta_L\beta_H}} = \frac{\sigma_{PH}}{\sqrt{\sigma_{PL}^2+2\sigma_{PL,PH}}} \tag{15.1}$$

Assuming $\sigma_{PL,PH} = 0$, then:

$$\frac{\beta_H}{\sqrt{\beta_L^2+2\beta_L\beta_H}} = \frac{\sigma_{PH}}{\sigma_{PL}} \tag{15.2}$$

If the ideal betas are the same as in example 13.1 ($\beta_L = 0.5$, $\beta_H = 1.5$) then the left-hand term of condition 2 is:

$$\frac{\beta_H}{\sqrt{\beta_L^2 + 2\beta_L\beta_H}} = \frac{\sigma_{PH}}{\sigma_{PL}} \quad \rightarrow \quad \frac{1.5}{\sqrt{0.5^2 + 2(0.5)(1.5)}} = 1.1339 = \frac{\sigma_{PH}}{\sigma_{PL}}$$

If $\sigma_{PH} = 10.6275\%$, and $\sigma_{PL} = 9.3725\%$, then condition 2 is met:

$$1.13389 = \frac{\sigma_{PH}}{\sigma_{PL}} = \frac{10.6274\%}{9.3725\%} = 1.13389$$

Because condition 2 is met, scenario 2 is expected.

If we take the numbers from this quick example and plug them into the formulas of videos 12 and 13, we get:

$$E(\hat{\alpha}_L') = -0.503\% \qquad E(\hat{\alpha}_H') = 0.503\%$$

$$E(\hat{S}_H') = 0.5170 \qquad E(\hat{S}_M') = 0.5170 \qquad E(\hat{S}_L') = 0.3848$$

Which means:

$$E(\hat{\alpha}_L') < 0 < E(\hat{\alpha}_H') \quad \text{and} \quad E(\hat{S}_H') = E(\hat{S}_M') > E(\hat{S}_L') \tag{14.2}$$

For the same average arithmetic total returns and ideal risks as found in example 13.1, expected alphas and expected Sharpe ratios can change in magnitude and order depending on the relationship between the perturbation risks.

Scenario 3

Now let’s look at Scenario 3:

$$E(\hat{\alpha}_L') < 0 < E(\hat{\alpha}_H') \quad \text{and} \quad E(\hat{S}_M') > E(\hat{S}_H') > E(\hat{S}_L') \tag{14.3}$$

Part of the derivation of the conditions of scenario 3 is identical to the derivation for the conditions of scenario 1, with the inequality reversed at every step.

$$\frac{\sigma_{PH}}{\sqrt{\sigma_{PL}^2+2\sigma_{PL,PH}}} > \frac{\beta_H}{\sqrt{\beta_L^2+2\beta_L\beta_H}} \tag{15.3}$$

Assuming $\sigma_{PL,PH} = 0$, then:

$$\frac{\sigma_{PH}}{\sigma_{PL}} > \frac{\beta_H}{\sqrt{\beta_L^2+2\beta_L\beta_H}} \tag{15.4}$$

This gives us the condition for when $E(\hat{S}'_M) > E(\hat{S}'_H)$.

Now we need the condition for when $E(\hat{S}'_H) > E(\hat{S}'_L)$.

Here is the derivation, beginning with the inequality:

$$E(\hat{S}'_H) > E(\hat{S}'_L) \quad \rightarrow \quad \frac{E(\alpha'_H)+\beta'_H(\bar{R}'_M-r_f)}{\sigma'_H} > \frac{E(\alpha'_L)+\beta'_L(\bar{R}'_M-r_f)}{\sigma'_L}$$

Making the following two substitutions:

$$E(\hat{\alpha}'_L) + \beta'_L(\bar{R}'_M - r_f) = \beta_L(\bar{R}'_M - r_f) \tag{12.5a}$$

$$E(\hat{\alpha}'_H) + \beta'_H(\bar{R}'_M - r_f) = \beta_H(\bar{R}'_M - r_f) \tag{12.5b}$$

And dividing by average excess return:

$$\frac{\beta_H(\bar{R}'_M-r_f)}{\sigma'_H} > \frac{\beta_L(\bar{R}'_M-r_f)}{\sigma'_L} \quad \rightarrow \quad \frac{\beta_H}{\sigma'_H} > \frac{\beta_L}{\sigma'_L}$$

Squaring both sides and expanding gives:

$$\frac{\beta_H^2}{\sigma_H'^2} > \frac{\beta_L^2}{\sigma_L'^2} \quad \rightarrow \quad \frac{\beta_H^2}{\sigma_H^2+\sigma_{PH}^2} > \frac{\beta_L^2}{\sigma_L^2+\sigma_{PL}^2} \quad \rightarrow \quad \frac{\beta_H^2}{\beta_H^2\sigma_M^2+\sigma_{PH}^2} > \frac{\beta_L^2}{\beta_L^2\sigma_M^2+\sigma_{PL}^2}$$

Taking the reciprocal of both sides and canceling:

$$\frac{\beta_H^2\sigma_M^2+\sigma_{PH}^2}{\beta_H^2} < \frac{\beta_L^2\sigma_M^2+\sigma_{PL}^2}{\beta_L^2} \quad \rightarrow \quad \sigma_M^2 + \frac{\sigma_{PH}^2}{\beta_H^2} < \sigma_M^2 + \frac{\sigma_{PL}^2}{\beta_L^2}$$

Subtracting like terms and moving things around:

$$\frac{\sigma_{PH}^2}{\beta_H^2} < \frac{\sigma_{PL}^2}{\beta_L^2} \quad \rightarrow \quad \frac{\sigma_{PH}^2}{\sigma_{PL}^2} < \frac{\beta_H^2}{\beta_L^2}$$

Taking the square root of both sides gives us:

$$\frac{\beta_H}{\beta_L} > \frac{\sigma_{PH}}{\sigma_{PL}}$$

Tentatively, the full condition for Scenario 3 is:

$$\frac{\beta_H}{\beta_L} > \frac{\sigma_{PH}}{\sigma_{PL}} > \frac{\beta_H}{\sqrt{\beta_L^2+2\beta_L\beta_H}} \tag{15.5}$$

This is too broad of a condition. While it gives the order of the expected Sharpe ratios, the order of the expected alphas changes within that range and we need to capture the point where the order changes. So I will leave the condition for Scenario 3 unfinished for the moment and move on to Scenario 4. Also, there will be numerical examples for all the scenarios in video 17. There is no need to continue doing numerical examples as we go.

Scenario 4

Let's look at Scenario 4, which is the scenario of the Capital Asset Pricing Model:

$$E(\hat{\alpha}'_L) = 0 = E(\hat{\alpha}'_H) \quad \text{and} \quad E(\hat{S}'_M) > E(\hat{S}'_H) > E(\hat{S}'_L) \tag{14.4}$$

The order of the expected Sharpe ratios is the same but now the expected alphas are equal. To get the same order of expected Sharpe ratios you need only review condition 3.

Now we have to find the condition for equivalent expected alphas, which means zero alphas. If the expected alphas are zero then the CAPM form as we know it, drops the alpha term, and is still equivalent to the ideal form.

$$E(\hat{\alpha}'_H) + \beta'_H(\bar{R}'_M - r_f) = \beta'_H(\bar{R}'_M - r_f) = \beta_H(\bar{R}'_M - r_f)$$

$$E(\hat{\alpha}'_L) + \beta'_L(\bar{R}'_M - r_f) = \beta'_L(\bar{R}'_M - r_f) = \beta_L(\bar{R}'_M - r_f)$$

Which means the dividing line between Scenario 3 and Scenario 5 is when ideal betas equal CAPM betas.

Which implies we can set the ratios between them equal, and expand the CAPM betas:

$$\frac{\beta_H}{\beta_L} = \frac{\beta'_H}{\beta'_L} \quad \rightarrow \quad \frac{\beta_H}{\beta_L} = \frac{\sigma'_{H,M}/\sigma'^2_M}{\sigma'_{L,M}/\sigma'^2_M} = \frac{\left(\beta_H\sigma^2_M + \frac{1}{2}\sigma^2_{PH} + \frac{1}{2}\sigma_{PL,PH}\right)}{\left(\beta_L\sigma^2_M + \frac{1}{2}\sigma^2_{PL} + \frac{1}{2}\sigma_{PL,PH}\right)}$$

Multiplying through by the denominator of the right-hand side:

$$\beta_H\left(\beta_L\sigma^2_M + \frac{1}{2}\sigma^2_{PL} + \frac{1}{2}\sigma_{PL,PH}\right) = \beta_L\left(\beta_H\sigma^2_M + \frac{1}{2}\sigma^2_{PH} + \frac{1}{2}\sigma_{PL,PH}\right)$$

Distributing the betas and subtracting like terms from both sides:

$$\frac{\beta_H}{2}\sigma^2_{PL} + \frac{\beta_H}{2}\sigma_{PL,PH} = \frac{\beta_L}{2}\sigma^2_{PH} + \frac{\beta_L}{2}\sigma_{PL,PH}$$

Multiplying through by 2, and moving terms around:

$$\frac{\beta_H}{\beta_L} = \frac{\sigma^2_{PH} + \sigma_{PL,PH}}{\sigma^2_{PL} + \sigma_{PL,PH}} \tag{15.6}$$

Assuming $\sigma_{PL,PH} = 0$, then:

$$\frac{\beta_H}{\beta_L} = \frac{\sigma^2_{PH}}{\sigma^2_{PL}} \tag{15.7}$$

Taking the square root of both sides gives the condition for scenario 4:

$$\sqrt{\frac{\beta_H}{\beta_L}} = \frac{\sigma_{PH}}{\sigma_{PL}} \tag{15.8}$$

Notice that this is not a range of values but an exact ratio, which is why CAPM is narrow and unlikely.

Now we also have the correct condition for Scenario 3.

$$\sqrt{\frac{\beta_H}{\beta_L}} > \frac{\sigma_{PH}}{\sigma_{PL}} > \frac{\beta_H}{\sqrt{\beta_L^2 + 2\beta_L\beta_H}} \tag{15.9}$$

Scenario 5

Next we derive the condition for scenario 5:

$$E(\hat{\alpha}_L') > 0 > E(\hat{\alpha}_H') \quad \text{and} \quad E(\hat{S}_M') > E(\hat{S}_H') > E(\hat{S}_L') \tag{14.5}$$

The condition for scenario 5 is available by inspection of the derivations of conditions 3 and 4.

It is:

$$\frac{\beta_H}{\beta_L} > \frac{\sigma_{PH}}{\sigma_{PL}} > \sqrt{\frac{\beta_H}{\beta_L}} \tag{15.10}$$

It is important to recognize that $E(\hat{\alpha}_L') > 0$ for this and all remaining scenarios.

This is enough for one video.

I will pick up in the next video by deriving the conditions for Scenarios 6 – 9.

At the end of the next video, there will be a summary table of the conditions needed for each scenario.

Alpha Found Video 16

More Detail Part 3

Welcome again to Alpha Found.

In this video we will finish deriving the conditions that lead to the 9 scenarios of expected outcomes.

Here are the 9 scenarios again (assuming $\bar{R}_{PM} = \bar{R}_{PL} = \bar{R}_{PH} = 0$).

Four scenarios contrary to the volatility effect:

1. $E(\hat{\alpha}'_L) < 0 < E(\hat{\alpha}'_H)$ and $E(\hat{S}'_H) > \boldsymbol{E(\hat{S}'_M)} > E(\hat{S}'_L)$ (14.1)

2. $E(\hat{\alpha}'_L) < 0 < E(\hat{\alpha}'_H)$ and $E(\hat{S}'_H) = \boldsymbol{E(\hat{S}'_M)} > E(\hat{S}'_L)$ (14.2)

3. $E(\hat{\alpha}'_L) < 0 < E(\hat{\alpha}'_H)$ and $\boldsymbol{E(\hat{S}'_M)} > E(\hat{S}'_H) > E(\hat{S}'_L)$ (14.3)

4. $E(\hat{\alpha}'_L) = 0 = E(\hat{\alpha}'_H)$ and $\boldsymbol{E(\hat{S}'_M)} > E(\hat{S}'_H) > E(\hat{S}'_L)$ (14.4)

Two scenarios weakly consistent with the volatility effect:

5. $E(\hat{\alpha}'_L) > 0 > E(\hat{\alpha}'_H)$ and $\boldsymbol{E(\hat{S}'_M)} > E(\hat{S}'_H) > E(\hat{S}'_L)$ (14.5)

6. $E(\hat{\alpha}'_L) > 0 > E(\hat{\alpha}'_H)$ and $\boldsymbol{E(\hat{S}'_M)} > E(\hat{S}'_H) = E(\hat{S}'_L)$ (14.6)

Three scenarios consistent with the volatility effect:

7. $E(\hat{\alpha}'_L) > 0 > E(\hat{\alpha}'_H)$ and $\boldsymbol{E(\hat{S}'_M)} > E(\hat{S}'_L) > E(\hat{S}'_H)$ (14.7)

8. $E(\hat{\alpha}'_L) > 0 > E(\hat{\alpha}'_H)$ and $\boldsymbol{E(\hat{S}'_M)} = E(\hat{S}'_L) > E(\hat{S}'_H)$ (14.8)

9. $E(\hat{\alpha}'_L) > 0 > E(\hat{\alpha}'_H)$ and $E(\hat{S}'_L) > \boldsymbol{E(\hat{S}'_M)} > E(\hat{S}'_H)$ (14.9)

Scenario 6

Now let's derive the conditions for Scenario 6:

$$E(\hat{\alpha}'_L) > 0 > E(\hat{\alpha}'_H) \quad \text{and} \quad E(\hat{S}'_M) > E(\hat{S}'_L) = E(\hat{S}'_H) \tag{14.6}$$

Note that $E(\hat{\alpha}'_L) > 0 > E(\hat{\alpha}'_H)$.

Part of condition 6 is derived the same as condition 5 with an equality in place of the inequality between the two left most terms of condition 5, such that:

$$\frac{\beta_H}{\beta_L} = \frac{\sigma_{PH}}{\sigma_{PL}} \tag{16.1}$$

Also, we need the condition:

$$E(\hat{S}'_M) > E(\hat{S}'_L) \quad \rightarrow \quad \frac{(\bar{R}'_M - r_f)}{\sigma'_M} > \frac{\alpha'_L + \beta'_L(\bar{R}'_M - r_f)}{\sigma'_L}$$

Since we can make this substitution:

$$E(\hat{\alpha}'_L) + \beta'_L(\bar{R}'_M - r_f) = \beta_L(\bar{R}'_M - r_f) \qquad (12.5a)$$

Then:

$$\frac{(\bar{R}'_M - r_f)}{\sigma'_M} > \frac{\beta_L(\bar{R}'_M - r_f)}{\sigma'_L} \quad \rightarrow \quad \frac{1}{\sigma'_M} > \frac{\beta_L}{\sigma'_L}$$

Squaring both sides, taking the reciprocal, and expanding gives this result:

$$\frac{1}{\sigma'^2_M} > \frac{\beta_L^2}{\sigma'^2_L} \quad \rightarrow \quad \sigma'^2_M < \frac{\sigma'^2_L}{\beta_L^2} \quad \rightarrow \quad \sigma_M^2 + \frac{\sigma_{PL}^2}{4} + \frac{\sigma_{PH}^2}{4} + \frac{\sigma_{PL,PH}}{2} < \frac{\beta_L^2\sigma_M^2 + \sigma_{PL}^2}{\beta_L^2}$$

Canceling β_L^2 on the right-hand side, and subtracting like terms:

$$\sigma_M^2 + \frac{\sigma_{PL}^2}{4} + \frac{\sigma_{PH}^2}{4} + \frac{\sigma_{PL,PH}}{2} < \sigma_M^2 + \frac{\sigma_{PL}^2}{\beta_L^2} \quad \rightarrow \quad \frac{\sigma_{PL}^2}{4} + \frac{\sigma_{PH}^2}{4} + \frac{\sigma_{PL,PH}}{2} < \frac{\sigma_{PL}^2}{\beta_L^2}$$

Multiplying through by 4 and again moving terms around leads to:

$$\sigma_{PH}^2 + 2\sigma_{PL,PH} < \left(\frac{4}{\beta_L^2} - 1\right)\sigma_{PL}^2$$

$$\frac{\sigma_{PH}^2 + 2\sigma_{PL,PH}}{\sigma_{PL}^2} < \frac{4 - \beta_L^2}{\beta_L^2} = \frac{\beta_H^2 + 2\beta_L\beta_H}{\beta_L^2}$$

Taking the square root of both sides, the full condition for scenario 6 is:

$$\frac{\sqrt{\beta_H^2 + 2\beta_L\beta_H}}{\beta_L} > \frac{\sqrt{\sigma_{PH}^2 + 2\sigma_{PL,PH}}}{\sigma_{PL}} = \frac{\beta_H}{\beta_L} \qquad (16.2)$$

Assuming $\sigma_{PL,PH} = 0$, the condition necessary for Scenario 6 is:

$$\frac{\sqrt{\beta_H^2 + 2\beta_L\beta_H}}{\beta_L} > \frac{\sigma_{PH}}{\sigma_{PL}} = \frac{\beta_H}{\beta_L} \qquad (16.3)$$

Scenario 7

Next is Scenario 7.

Scenario 7 is the same as scenario 6 with an inequality between the two right-hand terms of the Sharpe ratio relationships.

$$E(\hat{\alpha}'_L) > 0 > E(\hat{\alpha}'_H) \quad \text{and} \quad E(\hat{S}'_M) > E(\hat{S}'_L) > E(\hat{S}'_H) \qquad (14.7)$$

Therefore condition 7 is derived the same as condition 6 with an inequality between the two right-most terms:

$$\frac{\sqrt{\beta_H^2+2\beta_L\beta_H}}{\beta_L} > \frac{\sigma_{PH}}{\sigma_{PL}} > \frac{\beta_H}{\beta_L} \tag{16.4}$$

There will be a summary table showing all the conditions and scenarios in a few minutes.

Scenario 8

Now let's construct condition 8 for Scenario 8:

$$E(\hat{\alpha}_L') > 0 > E(\hat{\alpha}_H') \quad \text{and} \quad E(\hat{S}_M') = E(\hat{S}_L') > E(\hat{S}_H') \tag{14.8}$$

Scenario 8 is the same as Scenarios 6 and 7 with an equal sign between the two left-hand terms of the expected Sharpe ratio relationships. Therefore, condition 8 is derived the same as conditions 6 and 7 with an equal sign in place of the inequality in the two left-hand terms of the condition.

$$\frac{\sqrt{\beta_H^2+2\beta_L\beta_H}}{\beta_L} = \frac{\sigma_{PH}}{\sigma_{PL}} > \frac{\beta_H}{\beta_L} \tag{16.5}$$

Scenario 9

Finally let's look at Scenario 9:

$$E(\hat{\alpha}_L') > 0 > E(\hat{\alpha}_L') \quad \text{and} \quad E(\hat{S}_L') > E(\hat{S}_M') > E(\hat{S}_H') \tag{14.9}$$

Scenario 9 follows directly from Scenarios 7 and 8; by changing the direction of the inequality between the two left-hand terms of the expected Sharpe ratio relationships.

Therefore, condition 9 is derived the same as condition 7 and 8 however, there is only one inequality.

$$\frac{\sqrt{\sigma_{PH}^2+2\sigma_{PL,PH}}}{\sigma_{PL}} > \frac{\sqrt{\beta_H^2+2\beta_L\beta_H}}{\beta_L} \tag{16.6}$$

Assuming $\sigma_{PL,PH} = 0$, then:

$$\frac{\sigma_{PH}}{\sigma_{PL}} > \frac{\sqrt{\beta_H^2+2\beta_L\beta_H}}{\beta_L} \tag{16.7}$$

Putting all of the scenarios and conditions in one place gives us Table 16.1.

Table 16.1

	Expected $\widehat{\alpha}'$	Expected $\widehat{S}'$	Condition
1	$E(\hat{\alpha}'_L) < 0 < E(\hat{\alpha}'_H)$	$E(\hat{S}'_H) > E(\hat{S}'_M) > E(\hat{S}'_L)$	$\frac{\beta_H}{\sqrt{\beta_L^2+2\beta_L\beta_H}} > \frac{\sigma_{PH}}{\sigma_{PL}}$
2	$E(\hat{\alpha}'_L) < 0 < E(\hat{\alpha}'_H)$	$E(\hat{S}'_H) = E(\hat{S}'_M) > E(\hat{S}'_L)$	$\frac{\beta_H}{\sqrt{\beta_L^2+2\beta_L\beta_H}} = \frac{\sigma_{PH}}{\sigma_{PL}}$
3	$E(\hat{\alpha}'_L) < 0 < E(\hat{\alpha}'_H)$	$E(\hat{S}'_M) > E(\hat{S}'_H) > E(\hat{S}'_L)$	$\sqrt{\frac{\beta_H}{\beta_L}} > \frac{\sigma_{PH}}{\sigma_{PL}} > \frac{\beta_H}{\sqrt{\beta_L^2+2\beta_L\beta_H}}$
4	$E(\hat{\alpha}'_L) = 0 = E(\hat{\alpha}'_H)$	$E(\hat{S}'_M) > E(\hat{S}'_H) > E(\hat{S}'_L)$	$\sqrt{\frac{\beta_H}{\beta_L}} = \frac{\sigma_{PH}}{\sigma_{PL}}$
5	$E(\hat{\alpha}'_L) > 0 > E(\hat{\alpha}'_H)$	$E(\hat{S}'_M) > E(\hat{S}'_H) > E(\hat{S}'_L)$	$\frac{\beta_H}{\beta_L} > \frac{\sigma_{PH}}{\sigma_{PL}} > \sqrt{\frac{\beta_H}{\beta_L}}$
6	$E(\hat{\alpha}'_L) > 0 > E(\hat{\alpha}'_H)$	$E(\hat{S}'_M) > E(\hat{S}'_H) = E(\hat{S}'_L)$	$\frac{\sqrt{\beta_H^2+2\beta_L\beta_H}}{\beta_L} > \frac{\sigma_{PH}}{\sigma_{PL}} = \frac{\beta_H}{\beta_L}$
7	$E(\hat{\alpha}'_L) > 0 > E(\hat{\alpha}'_H)$	$E(\hat{S}'_M) > E(\hat{S}'_L) > E(\hat{S}'_H)$	$\frac{\sqrt{\beta_H^2+2\beta_L\beta_H}}{\beta_L} > \frac{\sigma_{PH}}{\sigma_{PL}} > \frac{\beta_H}{\beta_L}$
8	$E(\hat{\alpha}'_L) > 0 > E(\hat{\alpha}'_H)$	$E(\hat{S}'_M) = E(\hat{S}'_L) > E(\hat{S}'_H)$	$\frac{\sqrt{\beta_H^2+2\beta_L\beta_H}}{\beta_L} = \frac{\sigma_{PH}}{\sigma_{PL}} > \frac{\beta_H}{\beta_L}$
9	$E(\hat{\alpha}'_L) > 0 > E(\hat{\alpha}'_H)$	$E(\hat{S}'_L) > E(\hat{S}'_M) > E(\hat{S}'_H)$	$\frac{\sigma_{PH}}{\sigma_{PL}} > \frac{\sqrt{\beta_H^2+2\beta_L\beta_H}}{\beta_L}$

The scenarios and conditions have constant expected arithmetic mean return, constant ideal βs, and zero average perturbations during a generally rising market. If the condition is met, the associated scenario is expected. There is still a lot more to consider.

Alpha Found Video 17

Simulation Part 1: the setup

Welcome again to part 17.

This video is short. It sets up a simulation, with the results of the simulation in the next video.

With the nine scenarios defined, some attempt should be made to see if the conditions derived will add up to the predicted result.

Table 16.1

	Expected $\widehat{\alpha}'$	Expected $\widehat{S}'$	Condition
1	$E(\hat{\alpha}'_L) < 0 < E(\hat{\alpha}'_H)$	$E(\hat{S}'_H) > E(\hat{S}'_M) > E(\hat{S}'_L)$	$\frac{\beta_H}{\sqrt{\beta_L^2+2\beta_L\beta_H}} > \frac{\sigma_{PH}}{\sigma_{PL}}$
2	$E(\hat{\alpha}'_L) < 0 < E(\hat{\alpha}'_H)$	$E(\hat{S}'_H) = E(\hat{S}'_M) > E(\hat{S}'_L)$	$\frac{\beta_H}{\sqrt{\beta_L^2+2\beta_L\beta_H}} = \frac{\sigma_{PH}}{\sigma_{PL}}$
3	$E(\hat{\alpha}'_L) < 0 < E(\hat{\alpha}'_H)$	$E(\hat{S}'_M) > E(\hat{S}'_H) > E(\hat{S}'_L)$	$\sqrt{\frac{\beta_H}{\beta_L}} > \frac{\sigma_{PH}}{\sigma_{PL}} > \frac{\beta_H}{\sqrt{\beta_L^2+2\beta_L\beta_H}}$
4	$E(\hat{\alpha}'_L) = 0 = E(\hat{\alpha}'_H)$	$E(\hat{S}'_M) > E(\hat{S}'_H) > E(\hat{S}'_L)$	$\sqrt{\frac{\beta_H}{\beta_L}} = \frac{\sigma_{PH}}{\sigma_{PL}}$
5	$E(\hat{\alpha}'_L) > 0 > E(\hat{\alpha}'_H)$	$E(\hat{S}'_M) > E(\hat{S}'_H) > E(\hat{S}'_L)$	$\frac{\beta_H}{\beta_L} > \frac{\sigma_{PH}}{\sigma_{PL}} > \sqrt{\frac{\beta_H}{\beta_L}}$
6	$E(\hat{\alpha}'_L) > 0 > E(\hat{\alpha}'_H)$	$E(\hat{S}'_M) > E(\hat{S}'_H) = E(\hat{S}'_L)$	$\frac{\sqrt{\beta_H^2+2\beta_L\beta_H}}{\beta_L} > \frac{\sigma_{PH}}{\sigma_{PL}} = \frac{\beta_H}{\beta_L}$
7	$E(\hat{\alpha}'_L) > 0 > E(\hat{\alpha}'_H)$	$E(\hat{S}'_M) > E(\hat{S}'_L) > E(\hat{S}'_H)$	$\frac{\sqrt{\beta_H^2+2\beta_L\beta_H}}{\beta_L} > \frac{\sigma_{PH}}{\sigma_{PL}} > \frac{\beta_H}{\beta_L}$
8	$E(\hat{\alpha}'_L) > 0 > E(\hat{\alpha}'_H)$	$E(\hat{S}'_M) = E(\hat{S}'_L) > E(\hat{S}'_H)$	$\frac{\sqrt{\beta_H^2+2\beta_L\beta_H}}{\beta_L} = \frac{\sigma_{PH}}{\sigma_{PL}} > \frac{\beta_H}{\beta_L}$
9	$E(\hat{\alpha}'_L) > 0 > E(\hat{\alpha}'_H)$	$E(\hat{S}'_L) > E(\hat{S}'_M) > E(\hat{S}'_H)$	$\frac{\sigma_{PH}}{\sigma_{PL}} > \frac{\sqrt{\beta_H^2+2\beta_L\beta_H}}{\beta_L}$

To test the scenarios with simulation, the following procedure was used:

For each of the nine perturbation risk ratio scenarios, the same ideal market conditions were held constant. That is, for each perturbation risk ratio the following parameters from example 13.1 held:

The ideal betas were set as follows: $\beta_L = 0.5$ $\beta_H = 1.5$

Average excess return is: $\bar{R}_M - r_f = 10\%$

Average perturbation return is: $\bar{R}_M = 0\%$.

The ideal risks were set as seen here and in example 13.1:

$$\sigma_M = 18\% \qquad \sigma_L = \beta_L \sigma_M = 9\% \qquad \sigma_H = \beta_H \sigma_M = 27\%$$

The sum of the perturbation risks is set to 20%, which will be explained momentarily.

The specific levels of the perturbation risk ratio for each condition were set as follows:

Table 17.1
Simulation Conditions for Testing Table 16.1

	Condition	σ_{PH}/σ_{PL}	σ_{PH}	σ_{PL}
1	$\frac{\beta_H}{\sqrt{4-\beta_H^2}} > \frac{\sigma_{PH}}{\sigma_{PL}}$	1.00000	10.00000%	10.00000%
2	$\frac{\beta_H}{\sqrt{4-\beta_H^2}} = \frac{\sigma_{PH}}{\sigma_{PL}}$	1.13389	10.62746%	9.37253%
3	$\sqrt{\frac{\beta_H}{\beta_L}} > \frac{\sigma_{PH}}{\sigma_{PL}} > \frac{\beta_H}{\sqrt{4-\beta_H^2}}$	1.50000	12.00000%	8.00000%
4	$\sqrt{\frac{\beta_H}{\beta_L}} = \frac{\sigma_{PH}}{\sigma_{PL}}$	1.73205	12.67949%	7.32051%
5	$\frac{\beta_H}{\beta_L} > \frac{\sigma_{PH}}{\sigma_{PL}} > \sqrt{\frac{\beta_H}{\beta_L}}$	2.40000	14.11765%	5.88235%
6	$\frac{\beta_H}{\beta_L} = \frac{\sigma_{PH}}{\sigma_{PL}}$	3.00000	15.00000%	5.00000%
7	$\frac{\sqrt{4-\beta_L^2}}{\beta_L} > \frac{\sigma_{PH}}{\sigma_{PL}} > \frac{\beta_H}{\beta_L}$	3.44444	15.50000%	4.50000%
8	$\frac{\sqrt{4-\beta_L^2}}{\beta_L} = \frac{\sigma_{PH}}{\sigma_{PL}}$	3.87298	15.89574%	4.10426%
9	$\frac{\sigma_{PH}}{\sigma_{PL}} > \frac{\sqrt{4-\beta_L^2}}{\beta_L}$	5.00000	16.66666%	3.33333%

The specific levels were set to satisfy the nine conditions of the nine scenarios.

The conditions in Table 17.1 are the same as in Table 16.1, with the only difference being that specific numerical values are set for the perturbation risk parameters. Whenever there is an equal sign in the condition column, the value of the perturbation risk ratio is set by that condition, whenever there is a "greater than" sign, the value is set to be arbitrarily consistent with the condition.

Using Microsoft Excel®, excess returns for an ideal market portfolio were generated for 30 time periods using an expected mean excess return of 10%, and standard deviation of 18%.

This was performed for each of 30 time periods using the Excel® function shown here at the top of the page. Only three time periods are shown to demonstrate the method.

Norm.Inv(Rand(),10%,18%)

Period	**1**	**2**	**3**
Ideal high beta return	-10.22%	49.59%	7.12%
Ideal low beta return	-3.41%	16.53%	2.37%
Ideal market return	-6.81%	33.06%	4.75%
High perturbation return	11.34%	1.26%	-6.69%
Low perturbation return	-2.07%	-0.42%	-2.59%
Market perturbation return	4.64%	0.42%	-4.64%
High beta portfolio return	1.12%	50.85%	0.43%
Low beta portfolio return	-5.48%	16.11%	-0.22%
Market portfolio return	-2.18%	33.48%	0.11%

The returns for the ideal portion of the high and low $\hat{\beta}'$ portfolios were directly calculated by multiplying the ideal market return each period by its respective ideal β.

Similar to the random process that generated the ideal market returns, the series of perturbation returns with zero expected mean were randomly generated for each of the ideal portfolios.

The parameterized sum, which was set to 20%, was then divided up between the two perturbation risks in such a way that the parameterized ratio between them is set before the random process is carried out.

The two sets of perturbed portfolio returns were then added together to get a return series for the perturbed market portfolio, rebalancing after each period. Once the random perturbations were generated, they were added to their respective ideal portfolios as seen at the bottom of the screen.

Finally, 50,000 sets of simulated portfolio returns were run for each of nine conditions.

Now we need to look at the results of the 50,000 simulations for each condition, in the next video.

Alpha Found Video 18

Simulation Part 2

Welcome again to part 18 of Alpha Found.

This video is important to understanding the meaning of the perturbation risk model.

Although for now I am maintaining the assumption that $\sigma_{PL,PH} = 0$, the results of the simulation contain much insight. More important, the graphs of the results show us, or show me anyway, why alpha is so elusive.

Here are the fixed parameters of the simulations:

$$\beta_L = 0.5 \qquad \beta_H = 1.5$$

$$\bar{R}'_M - r_f = 10\% \qquad \bar{R}_{PM} = 0\%$$

$$\sigma_M = 18\% \qquad \sigma_L = \beta_L \sigma_M = 9\% \qquad \sigma_H = \beta_H \sigma_M = 27\%$$

$$\sigma_{PL} + \sigma_{PH} = 20.00\%$$

The only parameters that change from one set of conditions to the next are the perturbation risk parameters.

To begin, from the last video here are the conditions we are simulating:

Table 17.1
Simulation Conditions for Testing Table 16.1

	Condition	$\boldsymbol{\sigma_{PH}/\sigma_{PL}}$	$\boldsymbol{\sigma_{PH}}$	$\boldsymbol{\sigma_{PL}}$
1	$\frac{\beta_H}{\sqrt{4-\beta_H^2}} > \frac{\sigma_{PH}}{\sigma_{PL}}$	1.00000	10.00000%	10.00000%
2	$\frac{\beta_H}{\sqrt{4-\beta_H^2}} = \frac{\sigma_{PH}}{\sigma_{PL}}$	1.13389	10.62746%	9.37253%
3	$\sqrt{\frac{\beta_H}{\beta_L}} > \frac{\sigma_{PH}}{\sigma_{PL}} > \frac{\beta_H}{\sqrt{4-\beta_H^2}}$	1.50000	12.00000%	8.00000%
4	$\sqrt{\frac{\beta_H}{\beta_L}} = \frac{\sigma_{PH}}{\sigma_{PL}}$	1.73205	12.67949%	7.32051%
5	$\frac{\beta_H}{\beta_L} > \frac{\sigma_{PH}}{\sigma_{PL}} > \sqrt{\frac{\beta_H}{\beta_L}}$	2.40000	14.11765%	5.88235%
6	$\frac{\beta_H}{\beta_L} = \frac{\sigma_{PH}}{\sigma_{PL}}$	3.00000	15.00000%	5.00000%

7	$\frac{\sqrt{4-\beta_L^2}}{\beta_L} > \frac{\boldsymbol{\sigma_{PH}}}{\boldsymbol{\sigma_{PL}}} > \frac{\beta_H}{\beta_L}$	3.44444	15.50000%	4.50000%
8	$\frac{\sqrt{4-\beta_L^2}}{\beta_L} = \frac{\boldsymbol{\sigma_{PH}}}{\boldsymbol{\sigma_{PL}}}$	3.87298	15.89574%	4.10426%
9	$\frac{\boldsymbol{\sigma_{PH}}}{\boldsymbol{\sigma_{PL}}} > \frac{\sqrt{4-\beta_L^2}}{\beta_L}$	5.00000	16.66666%	3.33333%

Here are the numerical results of the simulation:

Table 18.1A

Expectation vs. Simulation Perturbation Risk Ratio Conditions 1 - 5

	1		2		3		4		5	
	$\sigma_{PH}/\sigma_{PL} = 1.0000$		$\sigma_{PH}/\sigma_{PL} = 1.1339$		$\sigma_{PH}/\sigma_{PL} = 1.5000$		$\sigma_{PH}/\sigma_{PL} = 1.7321$		$\sigma_{PH}/\sigma_{PL} = 2.4000$	
	Expected	**Simulated**	**Expected**	**Simulated**	**Expected**	**Simulated**	**Expected**	**Simulated**	**Expected**	**Simulated**
α'_H	0.668%	0.665%	0.503%	0.502%	0.160%	0.165%	0.000%	-0.004%	-0.312%	-0.310%
α'_L	-0.668%	-0.665%	-0.503%	-0.502%	-0.160%	-0.165%	0.000%	0.004%	0.312%	0.310%
β'_H	1.4332	1.4330	1.4497	1.4498	1.4840	1.4836	1.5000	1.5003	1.5312	1.5317
β'_L	0.5668	0.5670	0.5503	0.5502	0.5160	0.5164	0.5000	0.4997	0.4688	0.4683
S'_H	0.5210	0.5229	0.5170	0.5155	0.5077	0.5070	0.5029	0.5035	0.4923	0.4907
S'_L	0.3716	0.3728	0.3848	0.3837	0.4152	0.4142	0.4310	0.4319	0.4650	0.4640
S'_M	0.5171	0.5190	0.5170	0.5155	0.5157	0.5149	0.5146	0.5154	0.5113	0.5098
$\sigma'_{H,M}$	0.0536	0.0535	0.0542	0.0543	0.0558	0.0558	0.0566	0.0566	0.0586	0.0587
$\sigma'_{L,M}$	0.0212	0.0211	0.0206	0.0206	0.0194	0.0194	0.0189	0.0189	0.0179	0.0179
$\sigma'_{L,H}$	0.0243	0.0242	0.0243	0.0243	0.0243	0.0243	0.0243	0.0243	0.0243	0.0243
σ'_H	28.792%	28.759%	29.016%	29.020%	29.547%	29.532%	29.829%	29.832%	30.468%	30.495%
σ'_L	13.454%	13.445%	12.994%	12.995%	12.042%	12.048%	11.601%	11.594%	10.752%	10.750%
σ'_M	19.339%	19.315%	19.344%	19.347%	19.391%	19.386%	19.432%	19.430%	19.557%	19.571%
σ_{PH}	10.000%	10.006%	10.627%	10.638%	12.000%	11.993%	12.679%	12.672%	14.118%	14.117%
σ_{PL}	10.000%	9.999%	9.373%	9.370%	8.000%	8.003%	7.321%	7.320%	5.882%	5.877%
$\sigma_{PL,PH}$	0.000%	-0.001%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%
$\bar{R}'_H - r_f$	15.000%	15.037%	15.000%	14.959%	15.000%	14.972%	15.000%	15.022%	15.000%	14.965%
$\bar{R}'_L - r_f$	5.000%	5.012%	5.000%	4.986%	5.000%	4.991%	5.000%	5.007%	5.000%	4.988%
$\bar{R}'_M - r_f$	10.000%	10.025%	10.000%	9.973%	10.000%	9.981%	10.000%	10.015%	10.000%	9.976%

Table 18.1B

Expectation vs. Simulation Perturbation Risk Ratio Conditions 6 - 9

	6		7		8		9	
	$\sigma_{PH}/\sigma_{PL} = 3.0000$		$\sigma_{PH}/\sigma_{PL} = 3.4444$		$\sigma_{PH}/\sigma_{PL} = 3.8730$		$\sigma_{PH}/\sigma_{PL} = 5.0000$	
	Expected	Simulated	Expected	Simulated	Expected	Simulated	Expected	Simulated
α'_H	-0.485%	-0.490%	-0.577%	-0.578%	-0.646%	-0.648%	-0.771%	-0.765%
α'_L	0.485%	0.490%	0.577%	0.578%	0.646%	0.648%	0.771%	0.765%
β'_H	1.5485	1.5489	1.5577	1.5577	1.5646	1.5645	1.5771	1.5771
β'_L	0.4515	0.4511	0.4423	0.4423	0.4354	0.4355	0.4229	0.4229
S'_H	0.4856	0.4845	0.4818	0.4825	0.4787	0.4798	0.4727	0.4724
S'_L	0.4856	0.4848	0.4969	0.4977	0.5055	0.5065	0.5210	0.5206
S'_M	0.5087	0.5075	0.5069	0.5076	0.5055	0.5066	0.5024	0.5020
$\sigma'_{H,M}$	0.0599	0.0600	0.0606	0.0606	0.0612	0.0612	0.0625	0.0626
$\sigma'_{L,M}$	0.0175	0.0175	0.0172	0.0172	0.0170	0.0170	0.0168	0.0168
$\sigma'_{L,H}$	0.0243	0.0243	0.0243	0.0243	0.0243	0.0243	0.0243	0.0243
σ'_H	30.887%	30.921%	31.133%	31.123%	31.332%	31.315%	31.730%	31.750%
σ'_L	10.296%	10.300%	10.062%	10.056%	9.892%	9.887%	9.597%	9.603%
σ'_M	19.660%	19.678%	19.726%	19.719%	19.783%	19.772%	19.905%	19.918%
σ_{PH}	15.000%	15.007%	15.500%	15.495%	15.896%	15.892%	16.667%	16.689%
σ_{PL}	5.000%	5.000%	4.500%	4.493%	4.104%	4.100%	3.333%	3.332%
$\sigma_{PL,PH}$	0.000%	-0.001%	0.000%	0.000%	0.000%	-0.001%	0.000%	0.000%
$\bar{R}'_H - r_f$	15.000%	14.980%	15.000%	15.015%	15.000%	15.024%	15.000%	14.998%
$\bar{R}'_L - r_f$	5.000%	4.993%	5.000%	5.005%	5.000%	5.008%	5.000%	4.999%
$\bar{R}'_M - r_f$	10.000%	9.987%	10.000%	10.010%	10.000%	10.016%	10.000%	9.999%

It may be difficult or time consuming to read Table 18.1, but if you take the time, you will see how the various market statistics are just as predicted when the sample size is large. There really is not much worth talking about when comparing the expected values to the simulated values.

The real beauty of the simulation is in the graphs.

Figure 18.1
$E(\hat{\alpha}')$ as a function of the perturbation risk ratio for constant expected excess return set to 10%, constant ideal βs, and zero average perturbation return.

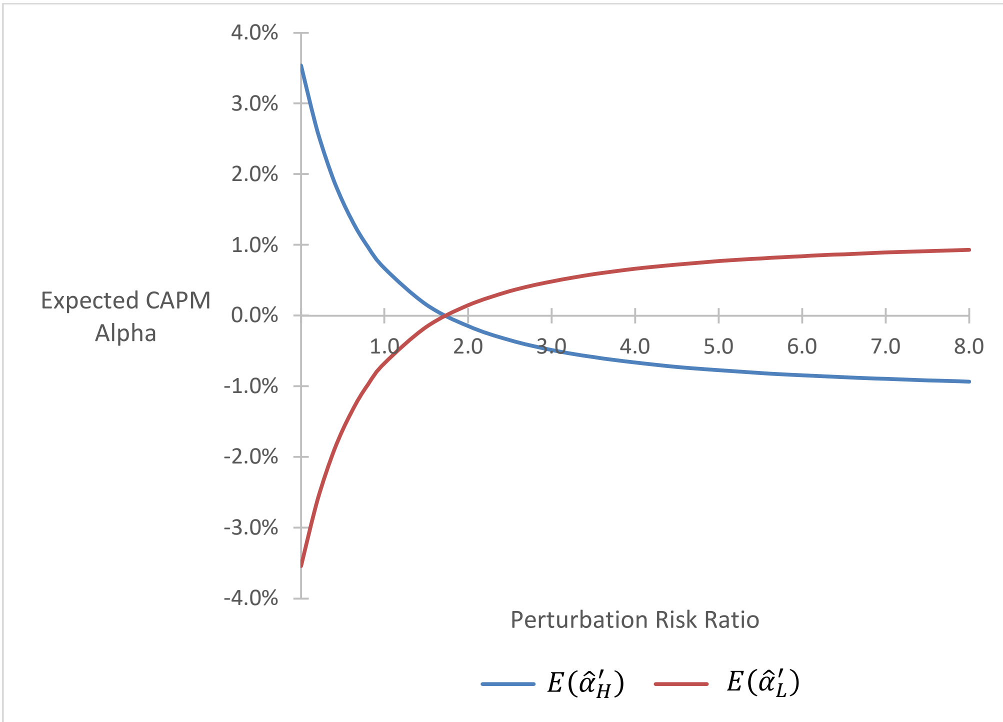


Figure 18.1 shows how $E(\hat{\alpha}')$ changes with the perturbation risk ratio while expected excess return does not change. Importantly, the changes in $E(\hat{\alpha}')$ are not linear with respect to the perturbation risk ratio. The rate of change for $E(\hat{\alpha}')$ flattens as the ratio increases. The one point of intersection occurs when the $E(\hat{\alpha}')$s are zero, which is scenario 4 (CAPM expectations).

In some ways, figure 18.1 is half the punchline of all the videos, since investment manager returns are evaluated on alpha. The blue curve is potential $E(\hat{\alpha}'_H)$ while expected excess return is fixed at 10%. The red curve is potential $E(\hat{\alpha}'_L)$ while expected excess market return is fixed at 10%. The different points on the curve are dependent on the perturbation risk ratio.

For any point on the blue curve representing an $E(\hat{\alpha}'_H)$, the corresponding red point on the same vertical line is the $E(\hat{\alpha}'_L)$ for the low CAPM beta portfolio.

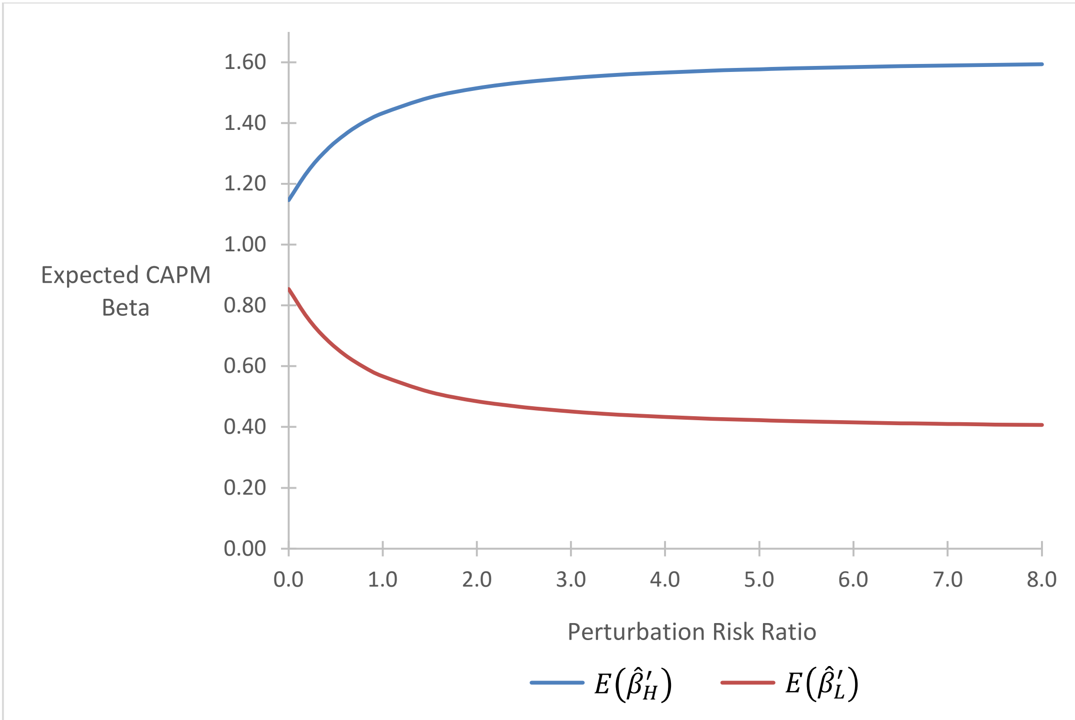
Alpha as measured by OLS is a point, but $E(\hat{\alpha}')$ is now seen to be a curve of potential values, each with the same realized arithmetic average return. That is why alpha is so elusive, $E(\hat{\alpha}')$ is not a point, it's a curve of potential expected values. The expected point on the potential expected alpha curve is determined by the PRR.

Since all the points on the $E(\hat{\alpha}')$ curve leave average return the same, it must mean that $E(\hat{\beta}')$ is also changing. That is $E(\hat{\alpha}')$ and $E(\hat{\beta}')$ move in concert such that the average arithmetic return is unchanged.

The next graph shows $E(\hat{\beta}')$. Like $E(\hat{\alpha}')$, $E(\hat{\beta}')$ is a curve of potential values.

Figure 18.2

$E(\hat{\beta}')$ as a function of the perturbation risk ratio for constant expected excess return set at 10%, constant ideal βs, and zero average perturbation return.



Just as with $E(\hat{\alpha}')$ in Figure 18.1, Figure 18.2 shows how $E(\hat{\beta}')$ changes with the perturbation risk ratio in a non-linear fashion while ideal β does not change. For any given market return and underlying risk parameters, $E(\hat{\alpha}')$ is determined by market return multiplied by the difference between ideal β and $E(\hat{\beta}')$, while $E(\hat{\beta}')$ is determined by ideal β and the PRR. (See equations 12.6a and 12.6b).

If the curve of potential $E(\hat{\alpha}')$ is half of the punchline, then the next figure is the remaining half of the punchline, since Sharpe ratios are the other major way investment managers are evaluated.

Figure 18.3

$E(\hat{S}')$ as a function of the perturbation risk ratio for constant expected excess return set to 10%, constant ideal βs, and zero average perturbation return.

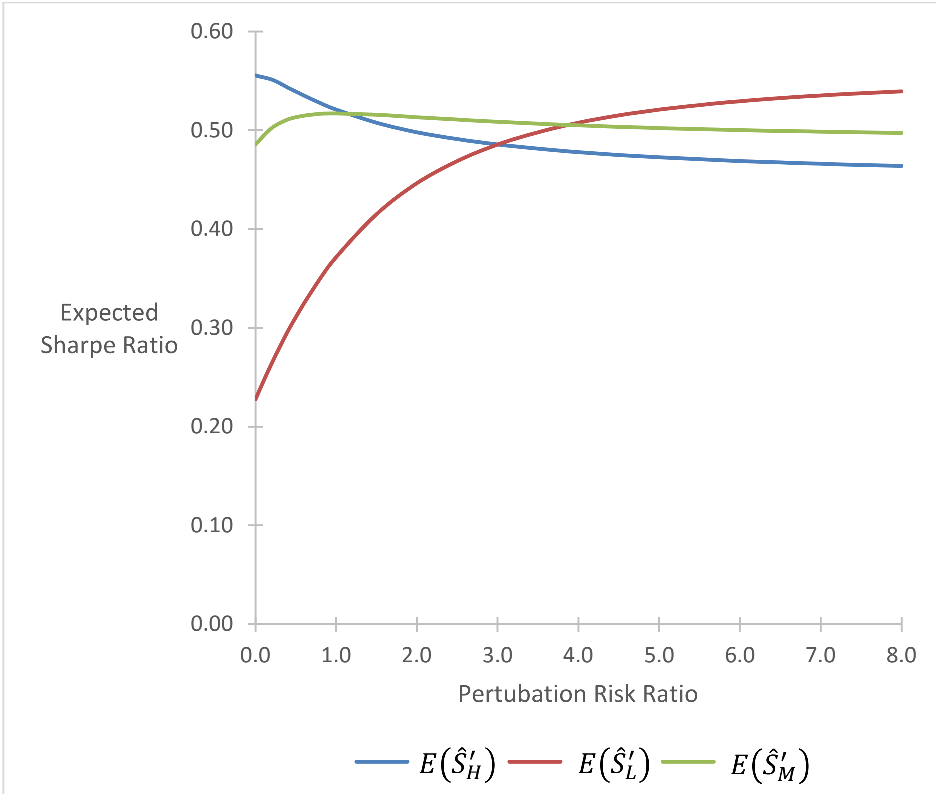


Figure 18.3 shows that the relationship between $E(\hat{S}')$ is far from obvious, while ideal Sharpe ratios do not change. The intersection on the left corresponds to scenario 2 (14.2), the middle intersection corresponds to scenario 6 (14.6), and the intersection on the right is scenario 8 (14.8). Anything to the right of scenario 8 corresponds to scenario 9.

It is not clear from a visual inspection of Figure 18.3, but the $E(\hat{S}_M')$ is at a maximum when the perturbation risk ratio equals 1, which is condition 1 in Tables 17.1 and 18.1.

The $E(\hat{S}_M')$ reaches its minimum at the two end points, when the perturbation risk ratio is zero on the far left, and also when the ratio is infinity on the far right.

To summarize, although the ideal parameters and expected market return is constant in the simulation, the various expected values of the statistics traditionally used to evaluate portfolio performance vary significantly depending on the perturbation risk ratio. That is, identical average portfolio performance can be evaluated differently due to events that do not affect arithmetic average return.

The consequences of the perturbation risk model are now on display. If all someone wanted to know is what it means, these graphs tell us what it means, mostly. However, the assumption of $\sigma_{PL,PH} = 0$, is still in play. Much more must be discussed in order to get underneath what is going on.

The next step is to make an effort to work the problem backward to see how different interpretations of the same data are possible. Working backward is more difficult, and the effort will ultimately be frustrated. As a result, we will be forced to let $\sigma_{PL,PH} \neq 0$.

There is still much more to come.

Alpha Found Video 19

The Backward Perturbation Risk Model: Parameter Estimates from Market Statistics

Welcome again to part 19 of Alpha Found.

The forward perturbation risk model is an ex-ante model. The backward perturbation risk model is an ex-post model. Given market statistics, we would want to know what ex-ante parameter values would most likely lead to market outcomes. Working backward from market data to underlying parameter estimates has significant challenges. However, starting with the knowledge gained from the forward model, some answers are available, while others are not.

Every parameter with a prime mark in the forward model has a corresponding parameter estimate originating from market data.

Each parameter estimate is directly analogous to the same formula for the related parameter as seen in prior videos.

$$\hat{\sigma}_L'^2 = \hat{\sigma}_L^2 + \hat{\sigma}_{PL}^2 = \hat{\beta}_L^2 \hat{\sigma}_M^2 + \hat{\sigma}_{PL}^2 \tag{19.1a}$$

$$\hat{\sigma}_H'^2 = \hat{\sigma}_H^2 + \hat{\sigma}_{PH}^2 = \hat{\beta}_H^2 \hat{\sigma}_M^2 + \hat{\sigma}_{PH}^2 \tag{19.1b}$$

$$\hat{\sigma}_M'^2 = \hat{\sigma}_M^2 + \frac{\hat{\sigma}_{PL}^2}{4} + \frac{\hat{\sigma}_{PH}^2}{4} \tag{19.2}$$

$$\hat{\sigma}_{L,M}' = \hat{\beta}_L \hat{\sigma}_M^2 + \frac{1}{2}\hat{\sigma}_{PL}^2 + \frac{1}{2}\hat{\sigma}_{PL,PH} \tag{19.3}$$

$$\hat{\sigma}_{L,H}' = \hat{\sigma}_{L,H} = \hat{\sigma}_L \hat{\sigma}_H = \hat{\beta}_L \hat{\beta}_H \hat{\sigma}_M^2 \tag{19.4}$$

The only difference between these new equations and the those in earlier videos is that the new ones are estimates, which come from data, and therefore they get a hat symbol to signify that they are estimates.

The following statistics are observable: $\hat{\sigma}_L'$, $\hat{\sigma}_M'$, $\hat{\sigma}_H'$, $\hat{\sigma}_{L,M}'$ and $\hat{\sigma}_{L,H}'$.

And these statistics are not observable: $\hat{\sigma}_M$, $\hat{\beta}_L$, $\hat{\beta}_H$, $\hat{\sigma}_{PL}$, and $\hat{\sigma}_{PH}$.

Once $\hat{\sigma}_{L,H}'$ (19.4) has been observed, a useful rearrangement can be made:

$$\hat{\sigma}_M^2 = \frac{\hat{\sigma}_{L,H}'}{\hat{\beta}_L \hat{\beta}_H} \tag{19.5}$$

Putting (19.5) into the ideal decomposition of variance gives:

$$\hat{\sigma}_L'^2 = \frac{\hat{\beta}_L \hat{\sigma}_{L,H}'}{\hat{\beta}_H} + \hat{\sigma}_{PL}^2 \tag{19.6a}$$

$$\hat{\sigma}_H'^2 = \frac{\hat{\beta}_H \hat{\sigma}_{L,H}'}{\hat{\beta}_L} + \hat{\sigma}_{PH}^2 \tag{19.6b}$$

Since $\hat{\beta}_L = 2 - \hat{\beta}_H$ then (19.6a) and (19.6b) have between them three unknowns in two equations.

There is not enough information to get an exact estimate of the ideal volatility parameters from the volatility statistics alone. However, constraints can be derived.

$\hat{\beta}_L$ is maximized when $\hat{\sigma}_{PL}^2 = 0$:

$$\hat{\sigma}_L'^2 = \hat{\beta}_L^2 \hat{\sigma}_M^2 + \hat{\sigma}_{PL}^2 = \frac{\hat{\beta}_L \hat{\sigma}'_{L,H}}{\hat{\beta}_H} + \hat{\sigma}_{PL}^2$$

If $\hat{\sigma}_{PL}^2 = 0$ then: $$\frac{\hat{\beta}_L \hat{\sigma}'_{L,H}}{\hat{\beta}_H} = \hat{\sigma}_L'^2$$

$$\hat{\beta}_L \hat{\sigma}'_{L,H} = \hat{\beta}_H \hat{\sigma}_L'^2 = \left(2 - \hat{\beta}_L\right) \hat{\sigma}_L'^2$$

Divide through by $\hat{\sigma}'_{L,H}$, and distribute $\left(2 - \hat{\beta}_L\right)$:

$$\hat{\beta}_L = \frac{2\hat{\sigma}_L'^2 - \hat{\beta}_L \hat{\sigma}_L'^2}{\hat{\sigma}'_{L,H}} = \frac{2\hat{\sigma}_L'^2}{\hat{\sigma}'_{L,H}} - \frac{\hat{\beta}_L \hat{\sigma}_L'^2}{\hat{\sigma}'_{L,H}}$$

Moving the right-hand term to the left side and moving things around gives:

$$\hat{\beta}_L + \frac{\hat{\beta}_L \hat{\sigma}_L'^2}{\hat{\sigma}'_{L,H}} = \frac{2\hat{\sigma}_L'^2}{\hat{\sigma}'_{L,H}} \quad \rightarrow \quad \hat{\beta}_L \left(1 + \frac{\hat{\sigma}_L'^2}{\hat{\sigma}'_{L,H}}\right) = \frac{2\hat{\sigma}_L'^2}{\hat{\sigma}'_{L,H}}$$

$$\rightarrow \quad \hat{\beta}_L \left(\frac{\hat{\sigma}'_{L,H} + \hat{\sigma}_L'^2}{\hat{\sigma}'_{L,H}}\right) = \frac{2\hat{\sigma}_L'^2}{\hat{\sigma}'_{L,H}} \quad \rightarrow$$

Divide through by $\left(\frac{\hat{\sigma}'_{L,H} + \hat{\sigma}_L'^2}{\hat{\sigma}'_{L,H}}\right)$ and cancel:

$$\hat{\beta}_L = \frac{2\hat{\sigma}_L'^2}{\hat{\sigma}'_{L,H}} \left(\frac{\hat{\sigma}'_{L,H}}{\hat{\sigma}'_{L,H} + \hat{\sigma}_L'^2}\right) = \left(\frac{2\hat{\sigma}_L'^2}{\hat{\sigma}'_{L,H} + \hat{\sigma}_L'^2}\right)$$

Therefore:

$$Max\left(\hat{\beta}_L\right) = \frac{2\hat{\sigma}_L'^2}{\hat{\sigma}'_{L,H} + \hat{\sigma}_L'^2} \tag{19.7}$$

Similarly, $\hat{\beta}_H$ is maximized when $\hat{\sigma}_{PH}^2 = 0$.

By a similar derivation we get:

$$\hat{\sigma}_H'^2 = \frac{\hat{\beta}_H \hat{\sigma}'_{L,H}}{\hat{\beta}_L} + \hat{\sigma}_{PH}^2 \quad \rightarrow \quad \frac{\hat{\beta}_H \hat{\sigma}'_{L,H}}{\hat{\beta}_L} = \hat{\sigma}_H'^2$$

$$\hat{\beta}_H \hat{\sigma}'_{L,H} = \hat{\beta}_L \hat{\sigma}_H'^2 = \left(2 - \hat{\beta}_H\right) \hat{\sigma}_H'^2$$

Divide through by $\hat{\sigma}'_{L,H}$ and distribute $\left(2 - \hat{\beta}_H\right)$:

$$\hat{\beta}_H = \frac{2\hat{\sigma}_H'^2 - \hat{\beta}_H \hat{\sigma}_H'^2}{\hat{\sigma}'_{L,H}} = \frac{2\hat{\sigma}_H'^2}{\hat{\sigma}'_{L,H}} - \frac{\hat{\beta}_H \hat{\sigma}_H'^2}{\hat{\sigma}'_{L,H}}$$

Moving the right-most term to the left side and moving things around give:

$$\hat{\beta}_H + \frac{\hat{\beta}_H \hat{\sigma}_H'^2}{\hat{\sigma}_{L,H}'} = \frac{2\hat{\sigma}_H'^2}{\hat{\sigma}_{L,H}'} \quad \rightarrow \quad \hat{\beta}_H \left(1 + \frac{\hat{\sigma}_H'^2}{\hat{\sigma}_{L,H}'}\right) = \frac{2\hat{\sigma}_H'^2}{\hat{\sigma}_{L,H}'}$$

$$\rightarrow \quad \hat{\beta}_H \left(\frac{\hat{\sigma}_{L,H}' + \hat{\sigma}_H'^2}{\hat{\sigma}_{L,H}'}\right) = \frac{2\hat{\sigma}_H'^2}{\hat{\sigma}_{L,H}'} \quad \rightarrow$$

Divide through by $\left(\frac{\hat{\sigma}_{L,H}' + \hat{\sigma}_H'^2}{\hat{\sigma}_{L,H}'}\right)$ and cancel:

$$\hat{\beta}_H = \frac{2\hat{\sigma}_H'^2}{\hat{\sigma}_{L,H}'} \left(\frac{\hat{\sigma}_{L,H}'}{\hat{\sigma}_{L,H}' + \hat{\sigma}_H'^2}\right) = \left(\frac{2\hat{\sigma}_H'^2}{\hat{\sigma}_{L,H}' + \hat{\sigma}_H'^2}\right)$$

$$Max(\hat{\beta}_H) = \frac{2\hat{\sigma}_H'^2}{\hat{\sigma}_{L,H}' + \hat{\sigma}_H'^2} \qquad (19.8)$$

If:

$$Min(\hat{\beta}_H) = 2 - Max(\hat{\beta}_L) = 2 - \frac{2\hat{\sigma}_L'^2}{\hat{\sigma}_{L,H}' + \hat{\sigma}_L'^2} = \frac{2\hat{\sigma}_{L,H}' + 2\hat{\sigma}_L'^2}{\hat{\sigma}_{L,H}' + \hat{\sigma}_L'^2} - \frac{2\hat{\sigma}_L'^2}{\hat{\sigma}_{L,H}' + \hat{\sigma}_L'^2}$$

Then:

$$Min(\hat{\beta}_H) = \frac{2\hat{\sigma}_{L,H}'}{\hat{\sigma}_{L,H}' + \hat{\sigma}_L'^2} \qquad (19.9)$$

Similarly:

$$Min(\hat{\beta}_L) = \frac{2\hat{\sigma}_{L,H}'}{\hat{\sigma}_{L,H}' + \hat{\sigma}_H'^2} \qquad (19.10)$$

$$Min(\hat{\beta}_L) \leq \hat{\beta}_L \leq Max(\hat{\beta}_L) \qquad (19.11)$$

The potential values for $\hat{\beta}_L$ between the minimum and maximum inclusive are continuous and correspond to different perturbation risks, different in their ratios and different in the sum of their magnitudes.

Since the range is continuous, only selected values can be evaluated. If the conclusions of example 13.1 are accepted as market statistics, a range of possible values for the ex-ante conditions can be derived.

Here are the statistics from example 13.1.

$$\hat{\sigma}_M'^2 = 0.039622 \qquad \hat{\sigma}_{L,H}' = 0.0243000$$

$$\hat{\sigma}_L'^2 = 0.009211 \qquad \hat{\sigma}_H'^2 = 0.1006777$$

$$\hat{\beta}_L' = 0.422882 \qquad \hat{\beta}_H' = 1.577117$$

$$\hat{\alpha}_L' = 0.77118\% \qquad \hat{\alpha}_H' = -0.77118\%$$

Using equations (19.7) and (19.8), the corresponding minimum and maximum ideal beta parameter estimates are:

$$Min(\hat{\beta}_L) = \frac{2\hat{\sigma}_{L,H}'}{\hat{\sigma}_{L,H}' + \hat{\sigma}_H'^2} = \frac{2(.0243)}{.02 \qquad (.10068)} = 0.38887$$

$$Max(\hat{\beta}_L) = \frac{2\hat{\sigma}_L'^2}{\hat{\sigma}_{L,H}' + \hat{\sigma}_L'^2} = \frac{2(.00921)}{.0243 + (.00921)} = 0.54974$$

For any given estimate of $\hat{\beta}_L$ between the values of 0.38887 and 0.54974 inclusive, an estimate of $\hat{\sigma}_{PL}^2$ follows. For example, given these parameter estimates:

$$\hat{\beta}_L = 0.38887 \qquad \hat{\beta}_H = 1.61113$$

$$\hat{\sigma}_L'^2 = 0.00921 \qquad \hat{\sigma}_{L,H} = 0.02430$$

then from equation (19.6a):

$$0.00921 = \frac{(0.388871)((0.02430)}{1.61113} + \hat{\sigma}_{PL}^2$$

$$\hat{\sigma}_{PL}^2 = 0.00334 \qquad \hat{\sigma}_{PL} = 0.05784$$

With the minimum and maximum low ideal beta estimates $(\hat{\beta}_L)$ in hand, any ideal beta estimate within that range has an associated pair of perturbations risks and a perturbation risk ratio that would generate the same CAPM $\hat{\beta}_L'$. Using (19.1a) and (19.1b) to estimate the perturbation risks ($\hat{\sigma}_{PL}^2$ and $\hat{\sigma}_{PH}^2$), for each $\hat{\beta}_L$ and $\hat{\beta}_H$ respectively, a table of potential parameter combinations is created. Each one is possibly the combination that accurately reflects the underlying reality.

Table 19.1

Relationship Between Ex-Post CAPM $\hat{\beta}'$, Estimated Ideal $\hat{\beta}$s, and Estimated Perturbation Risks as Derived from Example 13.1

Condition	$\hat{\beta}_L$	$\hat{\beta}_H$	$\hat{\sigma}_{PL}$	$\hat{\sigma}_{PH}$	$\hat{\sigma}_{PH}/\hat{\sigma}_{PL}$	*Notes*
1	0.3889	1.6111	0.05784	0.00000	0.00000	Minimum
1	0.4000	1.6000	0.05600	0.05897	1.05306	
2	0.4060	1.5940	0.05498	0.07255	1.31966	
3	0.4150	1.5850	0.05337	0.08871	1.66207	
4	0.4229	1.5771	0.05192	0.10026	1.93118	CAPM
5	0.4350	1.5650	0.04957	0.11512	2.32264	
6	0.4645	1.5355	0.04314	0.14262	3.30606	Equal Sharpe Ratios
7	0.4693	1.5307	0.04196	0.14635	3.48771	Midpoint
8	0.4822	1.5178	0.03863	0.15550	4.02571	
9	0.5000	1.5000	0.03333	0.16667	5.00000	No intercept model
9	0.5497	1.4503	0.00000	0.19124	Undef.	Maximum

Selected combinations of $\hat{\beta}_L$, $\hat{\beta}_H$, $\hat{\sigma}_{PL}$, and $\hat{\sigma}_{PH}$ are shown which result in the same realized portfolio returns, portfolio standard deviations, and portfolio CAPM $\hat{\beta}'$s. Which one reflects the true ex-ante conditions most accurately cannot be determined. The *Notes* column is explained in the examples given in the next video.

A subtle but important distinction needs to be emphasized. CAPM has a zero-intercept expectation under OLS. PRM leads to a non-zero intercept expectation under OLS. The "no intercept" method in the PRM is one of several methods of estimating ideal beta regardless of OLS expectations.

By inspection of Table 19.1, it takes only a small change in the estimated perturbation risks ($\hat{\sigma}_{PL}$ or $\hat{\sigma}_{PH}$) to generate a meaningful change in the relationship between expected Sharpe ratios. It would not take much for such a change to occur in the market.

You'll notice that there has been no perturbation α_P in this table or in any prior discussion, only a CAPM alpha. Perturbation α_P fits into the structure of the perturbation risk model just as α' does for CAPM, but perturbation α_P is not a straightforward measure of excess risk-adjusted return. It functions as an error term, but not in the usual sense. A later video derives and discusses the meaning of perturbation α_P in the context of the perturbation risk model. Consequently, there is no graph of estimated perturbation alphas.

Figure 19.1
Estimated ideal $\widehat{\boldsymbol{\beta}}$ as a function of the perturbation risk ratio for the given portfolio realized return and realized standard deviation.

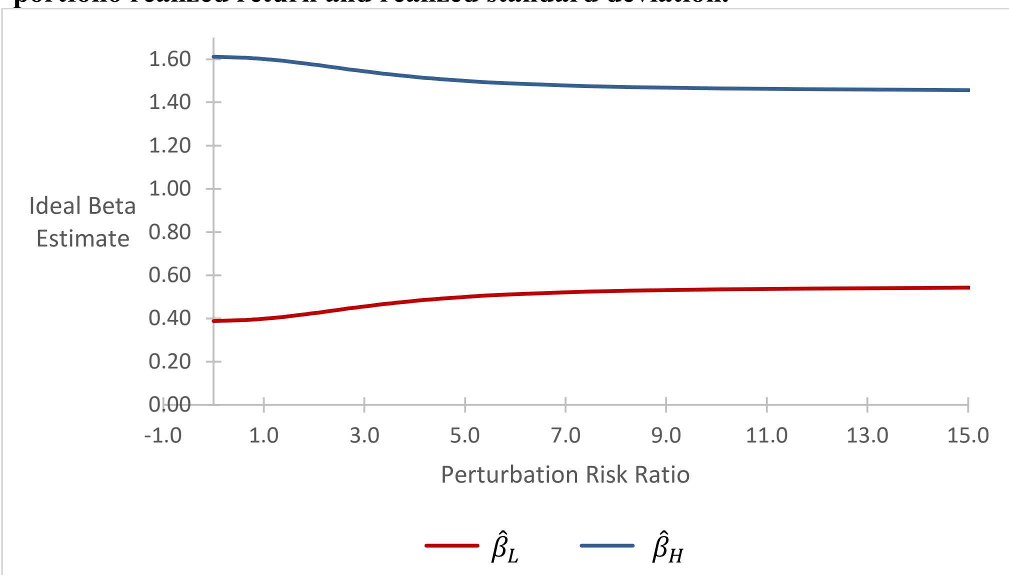
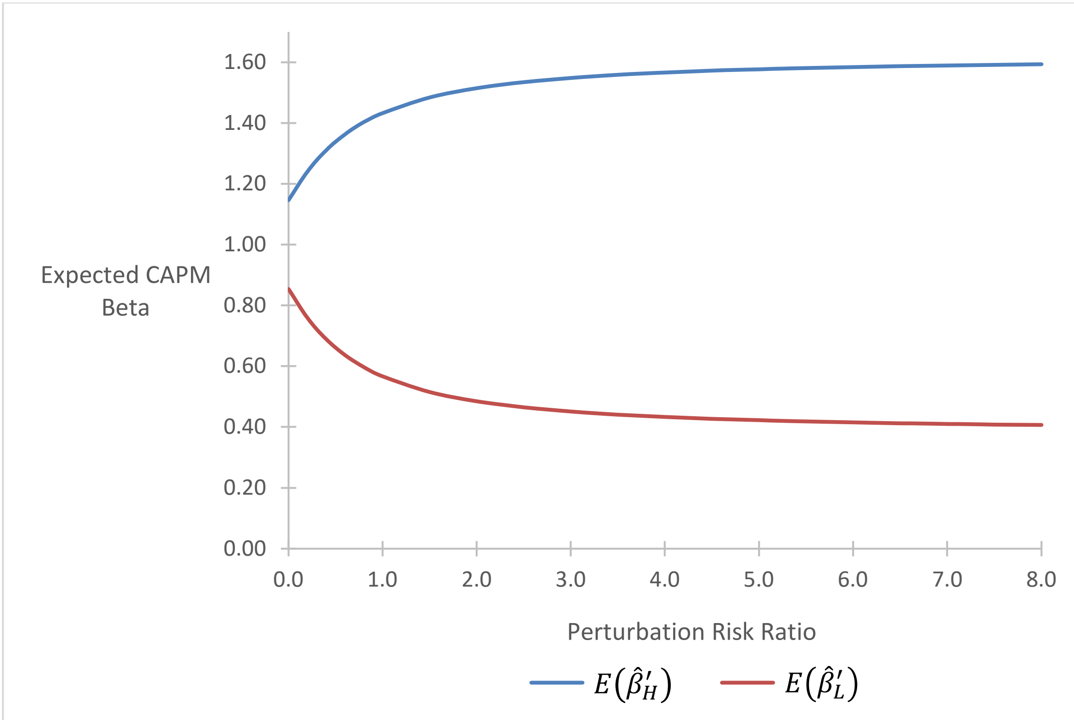


Figure 19.1 differs from Figure 18.2 in that Figure 19.1 starts with a $\hat{\beta}'$ and works toward $\hat{\beta}$ based on various perturbation risk ratios, while Figure 18.2 starts with an ideal β and works toward $E(\hat{\beta}')$. The different shapes of the curves between figures 18.2 and 19.1 come from the fact that they are inverse processes from one another.

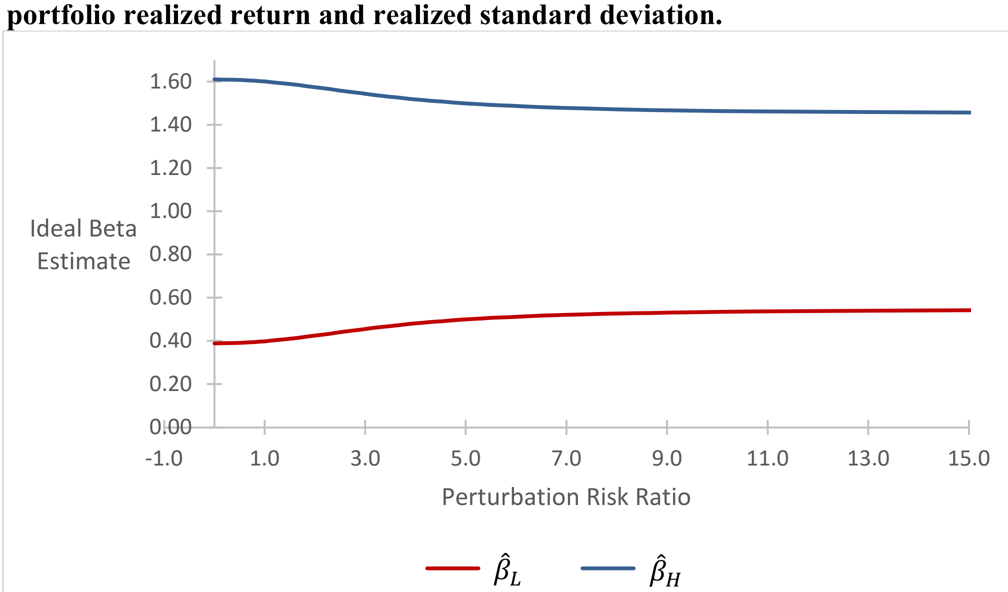
Here is Figure 18.2 for comparison.

Figure 18.2
$E(\widehat{\beta}')$ as a function of the perturbation risk ratio for constant expected excess return set at 10%, constant ideal βs, and zero average perturbation return.



And now back to Figure 19.1.

Figure 19.1
Estimated ideal $\widehat{\beta}$ as a function of the perturbation risk ratio for the given portfolio realized return and realized standard deviation.



Returning to Table 19.1

Table 19.1
Relationship Between Ex-Post CAPM $\widehat{\boldsymbol{\beta}}'$, Estimated Ideal $\widehat{\boldsymbol{\beta}}$s, and Estimated Perturbation Risks as Derived from Example 13.1

$Condition$	$\hat{\beta}_L$	$\hat{\beta}_H$	$\hat{\sigma}_{PL}$	$\hat{\sigma}_{PH}$	$\hat{\sigma}_{PH}/\hat{\sigma}_{PL}$	$Notes$
1	0.3889	1.6111	0.05784	0.00000	0.00000	Minimum
1	0.4000	1.6000	0.05600	0.05897	1.05306	
2	0.4060	1.5940	0.05498	0.07255	1.31966	
3	0.4150	1.5850	0.05337	0.08871	1.66207	
4	0.4229	1.5771	0.05192	0.10026	1.93118	CAPM with alpha
5	0.4350	1.5650	0.04957	0.11512	2.32264	
6	0.4645	1.5355	0.04314	0.14262	3.30606	Equal Sharpe Ratios
7	0.4693	1.5307	0.04196	0.14635	3.48771	Midpoint
8	0.4822	1.5178	0.03863	0.15550	4.02571	
9	0.5000	1.5000	0.03333	0.16667	5.00000	No intercept model
9	0.5497	1.4503	0.00000	0.19124	Undef.	Maximum

All of the possibilities in Table 19.1 have the same idiosyncratic risk.

Residual variance is unique in OLS in that only one line will minimize squared errors. However, only one of the possible perturbation risk combinations and related $\hat{\beta}$s is consistent with an OLS estimate of CAPM expectations (condition 4).

As difficult as it may seem to have an infinite number of risk combinations resulting in the same line, all from the same data, it results from the fact that in this example the explanatory variable is the sum of three random variables, ideal market returns plus two sets of perturbed returns, while the response variable is the sum of two random variables (ideal market returns scaled by ideal beta, plus one set of perturbed returns).

Once the data is fixed, the OLS is fixed, including residual variance. Yet there is more than one way to add up the component risks and returns to get the same answer. Greater ideal risk means less perturbation risk, and vice versa, but total risk remains unaltered.

Let me derive the equality of idiosyncratic risk for each entry of Table 19.1.

I'm going to start with the answer, that is the equation for idiosyncratic risk, and then derive it.

$$\hat{\sigma}_{\varepsilon}'^{2} = \left(\hat{\beta}_L - \hat{\beta}_L'\right)\left(\hat{\beta}_L\hat{\sigma}_M^2\right) + \left(1 - \frac{\hat{\beta}_L'}{2}\right)\hat{\sigma}_{PL}^2 \tag{19.12a}$$

$$\hat{\sigma}_{\varepsilon}'^{2} = \left(\hat{\beta}_H - \hat{\beta}_H'\right)\left(\hat{\beta}_H\hat{\sigma}_M^2\right) + \left(1 - \frac{\hat{\beta}_H'}{2}\right)\hat{\sigma}_{PH}^2 \tag{19.12b}$$

Setting the PRM decomposition of total risk equal to the CAPM decomposition gives us this:

$$\hat{\beta}_L^2\hat{\sigma}_M^2 + \hat{\sigma}_{PL}^2 = \hat{\beta}_L'^2\hat{\sigma}_M'^2 + \hat{\sigma}_{\varepsilon}'^2$$

$$\hat{\beta}_L^2\hat{\sigma}_M^2 + \hat{\sigma}_{PL}^2 = \hat{\beta}_L'\left(\hat{\beta}_L'\hat{\sigma}_M'^2\right) + \hat{\sigma}_{\varepsilon}'^2$$

By using the formula for low CAPM $\hat{\beta}_L'$ as a ratio of covariance to market variance:

$$\hat{\beta}_L^2\hat{\sigma}_M^2 + \hat{\sigma}_{PL}^2 = \hat{\beta}_L' \frac{\sigma_{L,M}'}{\sigma_M'^2}\hat{\sigma}_M'^2 + \hat{\sigma}_\varepsilon'^2 = \hat{\beta}_L'\sigma_{L,M}' + \hat{\sigma}_\varepsilon'^2$$

Substituting the PRM decomposition of $\hat{\sigma}_{L,M}'$ gives:

$$\hat{\beta}_L^2\hat{\sigma}_M^2 + \hat{\sigma}_{PL}^2 = \hat{\beta}_L'\left(\hat{\sigma}_L\hat{\sigma}_M + \frac{1}{2}\hat{\sigma}_{PL}^2 + \frac{1}{2}\hat{\sigma}_{PL,PH}\right) + \hat{\sigma}_\varepsilon'^2$$

Moving things around a bit gives us:

$$\hat{\beta}_L^2\hat{\sigma}_M^2 + \hat{\sigma}_{PL}^2 = \hat{\beta}_L'\hat{\beta}_L\hat{\sigma}_M^2 + \frac{\hat{\beta}_L'}{2}\hat{\sigma}_{PL}^2 + \frac{\hat{\beta}_L'}{2}\hat{\sigma}_{PL,PH} + \hat{\sigma}_\varepsilon'^2$$

$$\left(\hat{\beta}_L - \hat{\beta}_L'\right)\hat{\beta}_L\hat{\sigma}_M^2 + \sigma_{PL}^2 = \frac{\hat{\beta}_L'}{2}\hat{\sigma}_{PL}^2 + \frac{\hat{\beta}_L'}{2}\hat{\sigma}_{PL,PH} + \hat{\sigma}_\varepsilon'^2$$

Moving things around a bit more gives:

$$\left(\hat{\beta}_L - \hat{\beta}_L'\right)\hat{\beta}_L\hat{\sigma}_M^2 = \left(\frac{\hat{\beta}_L'}{2} - 1\right)\hat{\sigma}_{PL}^2 + \frac{\hat{\beta}_L'}{2}\hat{\sigma}_{PL,PH} + \hat{\sigma}_\varepsilon'^2$$

Finally:

$$\hat{\sigma}_\varepsilon'^2 = \left(\hat{\beta}_L - \hat{\beta}_L'\right)\hat{\beta}_L\hat{\sigma}_M^2 + \left(1 - \frac{\hat{\beta}_L'}{2}\right)\hat{\sigma}_{PL}^2 - \frac{\hat{\beta}_L'}{2}\hat{\sigma}_{PL,PH} \qquad 19.13a)$$

$$\hat{\sigma}_\varepsilon'^2 = \left(\hat{\beta}_H - \hat{\beta}_H'\right)\hat{\beta}_H\hat{\sigma}_M^2 + \left(1 - \frac{\hat{\beta}_H'}{2}\right)\hat{\sigma}_{PH}^2 - \frac{\hat{\beta}_H'}{2}\hat{\sigma}_{PL,PH} \qquad (19.13b)$$

Assuming $\hat{\sigma}_{PL,PH} = 0$, then:

$$\hat{\sigma}_\varepsilon'^2 = \left(\hat{\beta}_L - \hat{\beta}_L'\right)\hat{\beta}_L\hat{\sigma}_M^2 + \left(1 - \frac{\hat{\beta}_L'}{2}\right)\hat{\sigma}_{PL}^2 \qquad (19.12a)$$

$$\hat{\sigma}_\varepsilon'^2 = \left(\hat{\beta}_H - \hat{\beta}_H'\right)\hat{\beta}_H\hat{\sigma}_M^2 + \left(1 - \frac{\hat{\beta}_H'}{2}\right)\hat{\sigma}_{PH}^2 \qquad (19.12b)$$

You can now see with precision the differences between perturbation risk and idiosyncratic risk.

For calculation purposes, the estimate of ideal market risk (19.5) should be substituted into the equation for idiosyncratic risk (19.12a and b) because it's easier to work with, resulting in equations (19.14a and b).

$$\hat{\sigma}_\varepsilon'^2 = \left(\hat{\beta}_L - \hat{\beta}_L'\right)\left(\frac{\hat{\sigma}_{L,H}'}{\hat{\beta}_H}\right) + \left(1 - \frac{\hat{\beta}_L'}{2}\right)\hat{\sigma}_{PL}^2 \qquad (19.14a)$$

$$\hat{\sigma}_\varepsilon'^2 = \left(\hat{\beta}_H - \hat{\beta}_H'\right)\left(\frac{\hat{\sigma}_{L,H}'}{\hat{\beta}_L}\right) + \left(1 - \frac{\hat{\beta}_H'}{2}\right)\hat{\sigma}_{PH}^2 \qquad (19.14b)$$

The ideal betas and perturbation risks are different for each entry in Table 19.1, yet each entry has the same idiosyncratic risk. Worked examples demonstrating this fact are in the next video.

Alpha Found Video 19

The Backward Perturbation Risk Model: Parameter Estimates from Market Statistics

Welcome again to part 19 of Alpha Found.

The forward perturbation risk model is an ex-ante model. The backward perturbation risk model is an ex-post model. Given market statistics, we would want to know what ex-ante parameter values would most likely lead to market outcomes. Working backward from market data to underlying parameter estimates has significant challenges. However, starting with the knowledge gained from the forward model, some answers are available, while others are not.

Every parameter with a prime mark in the forward model has a corresponding parameter estimate originating from market data.

Each parameter estimate is directly analogous to the same formula for the related parameter as seen in prior videos.

$$\hat{\sigma}_L'^2 = \hat{\sigma}_L^2 + \hat{\sigma}_{PL}^2 = \hat{\beta}_L^2 \hat{\sigma}_M^2 + \hat{\sigma}_{PL}^2 \tag{19.1a}$$

$$\hat{\sigma}_H'^2 = \hat{\sigma}_H^2 + \hat{\sigma}_{PH}^2 = \hat{\beta}_H^2 \hat{\sigma}_M^2 + \hat{\sigma}_{PH}^2 \tag{19.1b}$$

$$\hat{\sigma}_M'^2 = \hat{\sigma}_M^2 + \frac{\hat{\sigma}_{PL}^2}{4} + \frac{\hat{\sigma}_{PH}^2}{4} \tag{19.2}$$

$$\hat{\sigma}_{L,M}' = \hat{\beta}_L \hat{\sigma}_M^2 + \frac{1}{2}\hat{\sigma}_{PL}^2 + \frac{1}{2}\hat{\sigma}_{PL,PH} \tag{19.3}$$

$$\hat{\sigma}_{L,H}' = \hat{\sigma}_{L,H} = \hat{\sigma}_L \hat{\sigma}_H = \hat{\beta}_L \hat{\beta}_H \hat{\sigma}_M^2 \tag{19.4}$$

The only difference between these new equations and the those in earlier videos is that the new ones are estimates, which come from data, and therefore they get a hat symbol to signify that they are estimates.

The following statistics are observable: $\hat{\sigma}_L'$, $\hat{\sigma}_M'$, $\hat{\sigma}_H'$, $\hat{\sigma}_{L,M}'$ and $\hat{\sigma}_{L,H}'$.

And these statistics are not observable: $\hat{\sigma}_M$, $\hat{\beta}_L$, $\hat{\beta}_H$, $\hat{\sigma}_{PL}$, and $\hat{\sigma}_{PH}$.

Once $\hat{\sigma}_{L,H}'$ (19.4) has been observed, a useful rearrangement can be made:

$$\hat{\sigma}_M^2 = \frac{\hat{\sigma}_{L,H}'}{\hat{\beta}_L \hat{\beta}_H} \tag{19.5}$$

Putting (19.5) into the ideal decomposition of variance gives:

$$\hat{\sigma}_L'^2 = \frac{\hat{\beta}_L \hat{\sigma}_{L,H}'}{\hat{\beta}_H} + \hat{\sigma}_{PL}^2 \tag{19.6a}$$

$$\hat{\sigma}_H'^2 = \frac{\hat{\beta}_H \hat{\sigma}_{L,H}'}{\hat{\beta}_L} + \hat{\sigma}_{PH}^2 \tag{19.6b}$$

Since $\hat{\beta}_L = 2 - \hat{\beta}_H$ then (19.6a) and (19.6b) have between them three unknowns in two equations.

There is not enough information to get an exact estimate of the ideal volatility parameters from the volatility statistics alone. However, constraints can be derived.

$\hat{\beta}_L$ is maximized when $\hat{\sigma}_{PL}^2 = 0$:

$$\hat{\sigma}_L'^2 = \hat{\beta}_L^2 \hat{\sigma}_M^2 + \hat{\sigma}_{PL}^2 = \frac{\hat{\beta}_L \hat{\sigma}'_{L,H}}{\hat{\beta}_H} + \hat{\sigma}_{PL}^2$$

If $\hat{\sigma}_{PL}^2 = 0$ then:

$$\frac{\hat{\beta}_L \hat{\sigma}'_{L,H}}{\hat{\beta}_H} = \hat{\sigma}_L'^2$$

$$\hat{\beta}_L \hat{\sigma}'_{L,H} = \hat{\beta}_H \hat{\sigma}_L'^2 = \left(2 - \hat{\beta}_L\right)\hat{\sigma}_L'^2$$

Divide through by $\hat{\sigma}'_{L,H}$, and distribute $\left(2 - \hat{\beta}_L\right)$:

$$\hat{\beta}_L = \frac{2\hat{\sigma}_L'^2 - \hat{\beta}_L \hat{\sigma}_L'^2}{\hat{\sigma}'_{L,H}} = \frac{2\hat{\sigma}_L'^2}{\hat{\sigma}'_{L,H}} - \frac{\hat{\beta}_L \hat{\sigma}_L'^2}{\hat{\sigma}'_{L,H}}$$

Moving the right-hand term to the left side and moving things around gives:

$$\hat{\beta}_L + \frac{\hat{\beta}_L \hat{\sigma}_L'^2}{\hat{\sigma}'_{L,H}} = \frac{2\hat{\sigma}_L'^2}{\hat{\sigma}'_{L,H}} \quad \rightarrow \quad \hat{\beta}_L\left(1 + \frac{\hat{\sigma}_L'^2}{\hat{\sigma}'_{L,H}}\right) = \frac{2\hat{\sigma}_L'^2}{\hat{\sigma}'_{L,H}}$$

$$\rightarrow \quad \hat{\beta}_L\left(\frac{\hat{\sigma}'_{L,H} + \hat{\sigma}_L'^2}{\hat{\sigma}'_{L,H}}\right) = \frac{2\hat{\sigma}_L'^2}{\hat{\sigma}'_{L,H}} \quad \rightarrow$$

Divide through by $\left(\frac{\hat{\sigma}'_{L,H} + \hat{\sigma}_L'^2}{\hat{\sigma}'_{L,H}}\right)$ and cancel:

$$\hat{\beta}_L = \frac{2\hat{\sigma}_L'^2}{\hat{\sigma}'_{L,H}}\left(\frac{\hat{\sigma}'_{L,H}}{\hat{\sigma}'_{L,H} + \hat{\sigma}_L'^2}\right) = \left(\frac{2\hat{\sigma}_L'^2}{\hat{\sigma}'_{L,H} + \hat{\sigma}_L'^2}\right)$$

Therefore:

$$Max\left(\hat{\beta}_L\right) = \frac{2\hat{\sigma}_L'^2}{\hat{\sigma}'_{L,H} + \hat{\sigma}_L'^2} \tag{19.7}$$

Similarly, $\hat{\beta}_H$ is maximized when $\hat{\sigma}_{PH}^2 = 0$.

By a similar derivation we get:

$$\hat{\sigma}_H'^2 = \frac{\hat{\beta}_H \hat{\sigma}'_{L,H}}{\hat{\beta}_L} + \hat{\sigma}_{PH}^2 \quad \rightarrow \quad \frac{\hat{\beta}_H \hat{\sigma}'_{L,H}}{\hat{\beta}_L} = \hat{\sigma}_H'^2$$

$$\hat{\beta}_H \hat{\sigma}'_{L,H} = \hat{\beta}_L \hat{\sigma}_H'^2 = \left(2 - \hat{\beta}_H\right)\hat{\sigma}_H'^2$$

Divide through by $\hat{\sigma}'_{L,H}$ and distribute $\left(2 - \hat{\beta}_H\right)$:

$$\hat{\beta}_H = \frac{2\hat{\sigma}_H'^2 - \hat{\beta}_H \hat{\sigma}_H'^2}{\hat{\sigma}'_{L,H}} = \frac{2\hat{\sigma}_H'^2}{\hat{\sigma}'_{L,H}} - \frac{\hat{\beta}_H \hat{\sigma}_H'^2}{\hat{\sigma}'_{L,H}}$$

Moving the right-most term to the left side and moving things around give:

$$\hat{\beta}_H + \frac{\hat{\beta}_H \hat{\sigma}_H'^2}{\hat{\sigma}_{L,H}'} = \frac{2\hat{\sigma}_H'^2}{\hat{\sigma}_{L,H}'} \quad \rightarrow \quad \hat{\beta}_H \left(1 + \frac{\hat{\sigma}_H'^2}{\hat{\sigma}_{L,H}'}\right) = \frac{2\hat{\sigma}_H'^2}{\hat{\sigma}_{L,H}'}$$

$$\rightarrow \quad \hat{\beta}_H \left(\frac{\hat{\sigma}_{L,H}' + \hat{\sigma}_H'^2}{\hat{\sigma}_{L,H}'}\right) = \frac{2\hat{\sigma}_H'^2}{\hat{\sigma}_{L,H}'} \quad \rightarrow$$

Divide through by $\left(\frac{\hat{\sigma}_{L,H}' + \hat{\sigma}_H'^2}{\hat{\sigma}_{L,H}'}\right)$ and cancel:

$$\hat{\beta}_H = \frac{2\hat{\sigma}_H'^2}{\hat{\sigma}_{L,H}'}\left(\frac{\hat{\sigma}_{L,H}'}{\hat{\sigma}_{L,H}' + \hat{\sigma}_H'^2}\right) = \left(\frac{2\hat{\sigma}_H'^2}{\hat{\sigma}_{L,H}' + \hat{\sigma}_H'^2}\right)$$

$$Max(\hat{\beta}_H) = \frac{2\hat{\sigma}_H'^2}{\hat{\sigma}_{L,H}' + \hat{\sigma}_H'^2} \tag{19.8}$$

If:

$$Min(\hat{\beta}_H) = 2 - Max(\hat{\beta}_L) = 2 - \frac{2\hat{\sigma}_L'^2}{\hat{\sigma}_{L,H}' + \hat{\sigma}_L'^2} = \frac{2\hat{\sigma}_{L,H}' + 2\hat{\sigma}_L'^2}{\hat{\sigma}_{L,H}' + \hat{\sigma}_L'^2} - \frac{2\hat{\sigma}_L'^2}{\hat{\sigma}_{L,H}' + \hat{\sigma}_L'^2}$$

Then:

$$Min(\hat{\beta}_H) = \frac{2\hat{\sigma}_{L,H}'}{\hat{\sigma}_{L,H}' + \hat{\sigma}_L'^2} \tag{19.9}$$

Similarly:

$$Min(\hat{\beta}_L) = \frac{2\hat{\sigma}_{L,H}'}{\hat{\sigma}_{L,H}' + \hat{\sigma}_H'^2} \tag{19.10}$$

$$Min(\hat{\beta}_L) \le \hat{\beta}_L \le Max(\hat{\beta}_L) \tag{19.11}$$

The potential values for $\hat{\beta}_L$ between the minimum and maximum inclusive are continuous and correspond to different perturbation risks, different in their ratios and different in the sum of their magnitudes.

Since the range is continuous, only selected values can be evaluated. If the conclusions of example 13.1 are accepted as market statistics, a range of possible values for the ex-ante conditions can be derived.

Here are the statistics from example 13.1.

$$\hat{\sigma}_M'^2 = 0.039622 \qquad \hat{\sigma}_{L,H}' = 0.0243000$$

$$\hat{\sigma}_L'^2 = 0.009211 \qquad \hat{\sigma}_H'^2 = 0.1006777$$

$$\hat{\beta}_L' = 0.422882 \qquad \hat{\beta}_H' = 1.577117$$

$$\hat{\alpha}_L' = 0.77118\% \qquad \hat{\alpha}_H' = -0.77118\%$$

Using equaitons (19.7) and (19.8), the corresponding minimum and maximum ideal beta parameter estimates are:

$$Min(\hat{\beta}_L) = \frac{2\hat{\sigma}_{L,H}'}{\hat{\sigma}_{L,H}' + \hat{\sigma}_H'^2} = \frac{2(.0243)}{.0243 + (.10068)} = 0.38887$$

$$Max(\hat{\beta}_L) = \frac{2\hat{\sigma}_L'^2}{\hat{\sigma}_{L,H}' + \hat{\sigma}_L'^2} = \frac{2(.00921)}{.0243 + (.00921)} = 0.54974$$

For any given estimate of $\hat{\beta}_L$ between the values of 0.38887 and 0.54974 inclusive, an estimate of $\hat{\sigma}_{PL}^2$ follows. For example, given these parameter estimates:

$$\hat{\beta}_L = 0.38887 \qquad \hat{\beta}_H = 1.61113$$

$$\hat{\sigma}_L'^2 = 0.00921 \qquad \hat{\sigma}_{L,H} = 0.02430$$

then from equation (19.6a):

$$0.00921 = \frac{(0.388871)((0.02430)}{1.61113} + \hat{\sigma}_{PL}^2$$

$$\hat{\sigma}_{PL}^2 = 0.00334 \qquad \hat{\sigma}_{PL} = 0.05784$$

With the minimum and maximum low ideal beta estimates $(\hat{\beta}_L)$ in hand, any ideal beta estimate within that range has an associated pair of perturbations risks and a perturbation risk ratio that would generate the same CAPM $\hat{\beta}_L'$. Using (19.1a) and (19.1b) to estimate the perturbation risks ($\hat{\sigma}_{PL}^2$ and $\hat{\sigma}_{PH}^2$), for each $\hat{\beta}_L$ and $\hat{\beta}_H$ respectively, a table of potential parameter combinations is created. Each one is possibly the combination that accurately reflects the underlying reality.

Table 19.1
Relationship Between Ex-Post CAPM $\hat{\beta}'$, Estimated Ideal $\hat{\beta}$s, and Estimated Perturbation Risks as Derived from Example 13.1

$Condition$	$\hat{\beta}_L$	$\hat{\beta}_H$	$\hat{\sigma}_{PL}$	$\hat{\sigma}_{PH}$	$\hat{\sigma}_{PH}/\hat{\sigma}_{PL}$	$Notes$
1	0.3889	1.6111	0.05784	0.00000	0.00000	Minimum
1	0.4000	1.6000	0.05600	0.05897	1.05306	
2	0.4060	1.5940	0.05498	0.07255	1.31966	
3	0.4150	1.5850	0.05337	0.08871	1.66207	
4	0.4229	1.5771	0.05192	0.10026	1.93118	CAPM
5	0.4350	1.5650	0.04957	0.11512	2.32264	
6	0.4645	1.5355	0.04314	0.14262	3.30606	Equal Sharpe Ratios
7	0.4693	1.5307	0.04196	0.14635	3.48771	Midpoint
8	0.4822	1.5178	0.03863	0.15550	4.02571	
9	0.5000	1.5000	0.03333	0.16667	5.00000	No intercept model
9	0.5497	1.4503	0.00000	0.19124	Undef.	Maximum

Selected combinations of $\hat{\beta}_L$, $\hat{\beta}_H$, $\hat{\sigma}_{PL}$, and $\hat{\sigma}_{PH}$ are shown which result in the same realized portfolio returns, portfolio standard deviations, and portfolio CAPM $\hat{\beta}'$s. Which one reflects the true ex-ante conditions most accurately cannot be determined. The *Notes* column is explained in the examples given in the next video.

A subtle but important distinction needs to be emphasized. CAPM has a zero-intercept expectation under OLS. PRM leads to a non-zero intercept expectation under OLS. The "no intercept" method in the PRM is one of several methods of estimating ideal beta regardless of OLS expectations.

By inspection of Table 19.1, it takes only a small change in the estimated perturbation risks ($\hat{\sigma}_{PL}$ or $\hat{\sigma}_{PH}$) to generate a meaningful change in the relationship between expected Sharpe ratios. It would not take much for such a change to occur in the market.

You'll notice that there has been no perturbation α_P in this table or in any prior discussion, only a CAPM alpha. Perturbation α_P fits into the structure of the perturbation risk model just as α' does for CAPM, but perturbation α_P is not a straightforward measure of excess risk-adjusted return. It functions as an error term, but not in the usual sense. A later video derives and discusses the meaning of perturbation α_P in the context of the perturbation risk model. Consequently, there is no graph of estimated perturbation alphas.

Figure 19.1
Estimated ideal $\hat{\beta}$ as a function of the perturbation risk ratio for the given portfolio realized return and realized standard deviation.

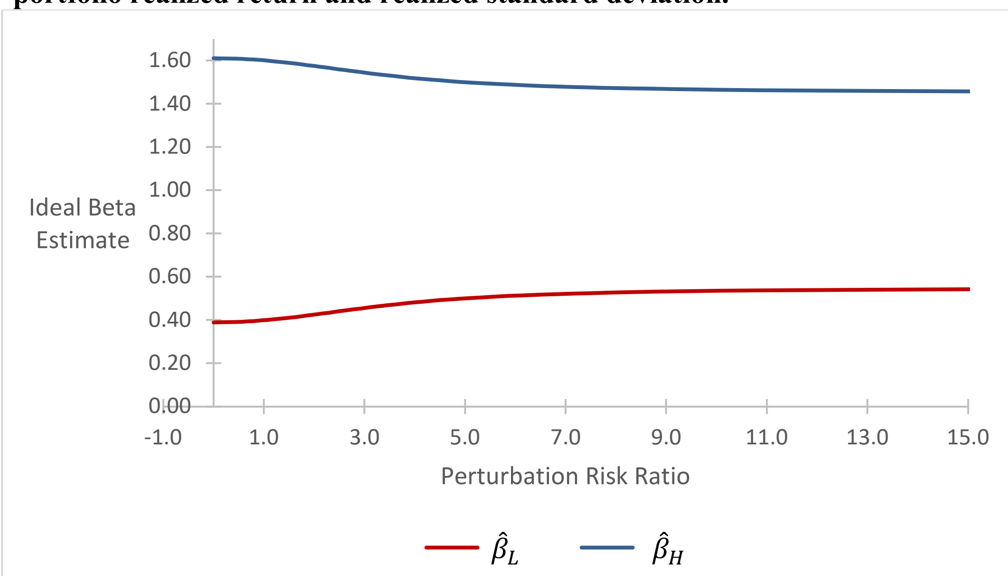
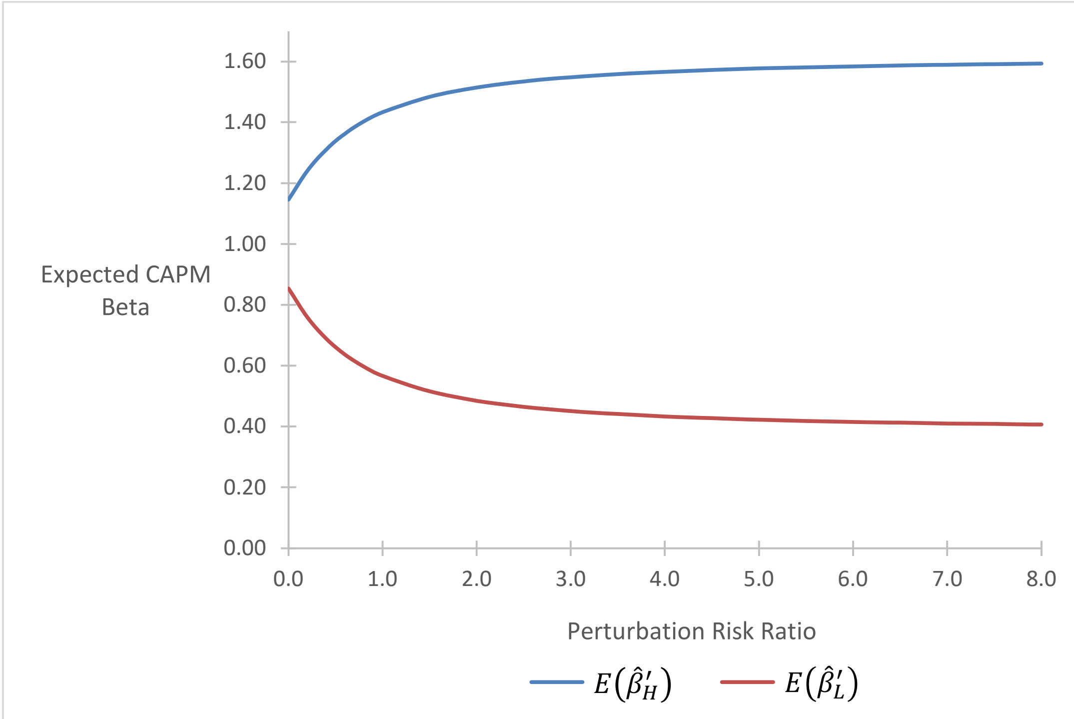


Figure 19.1 differs from Figure 18.2 in that Figure 19.1 starts with a $\hat{\beta}'$ and works toward $\hat{\beta}$ based on various perturbation risk ratios, while Figure 18.2 starts with an ideal β and works toward $E(\hat{\beta}')$. The different shapes of the curves between figures 18.2 and 19.1 come from the fact that they are inverse processes from one another.

Here is Figure 18.2 for comparison.

Figure 18.2

$E(\widehat{\beta}')$ as a function of the perturbation risk ratio for constant expected excess return set at 10%, constant ideal βs, and zero average perturbation return.



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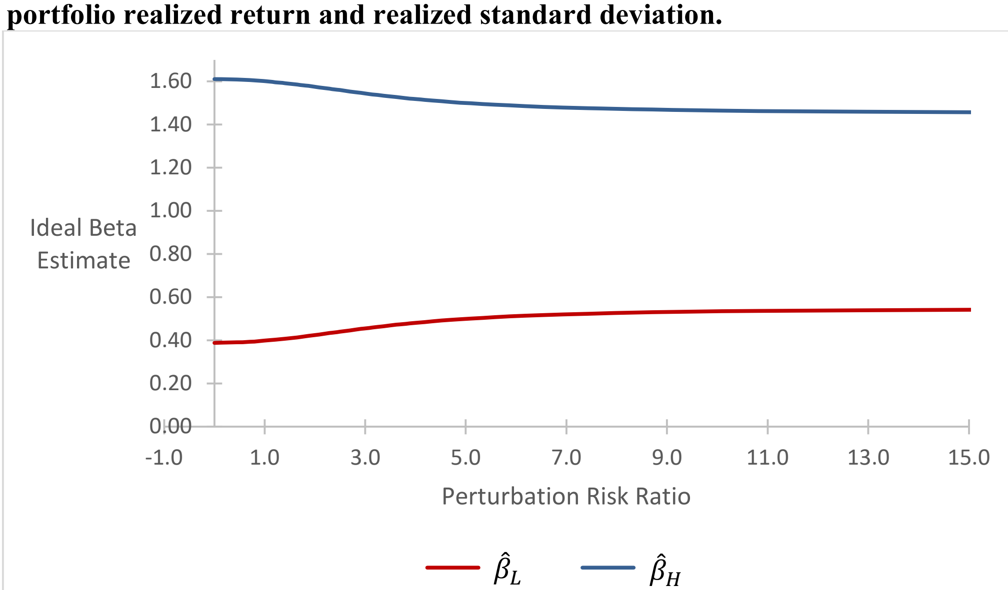

Returning to Table 19.1

Table 19.1

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All of the possibilities in Table 19.1 have the same idiosyncratic risk.

Residual variance is unique in OLS in that only one line will minimize squared errors. However, only one of the possible perturbation risk combinations and related $\hat{\beta}$s is consistent with an OLS estimate of CAPM expectations (condition 4).

As difficult as it may seem to have an infinite number of risk combinations resulting in the same line, all from the same data, it results from the fact that in this example the explanatory variable is the sum of three random variables, ideal market returns plus two sets of perturbed returns, while the response variable is the sum of two random variables (ideal market returns scaled by ideal beta, plus one set of perturbed returns).

Once the data is fixed, the OLS is fixed, including residual variance. Yet there is more than one way to add up the component risks and returns to get the same answer. Greater ideal risk means less perturbation risk, and vice versa, but total risk remains unaltered.

Let me derive the equality of idiosyncratic risk for each entry of Table 19.1.

I'm going to start with the answer, that is the equation for idiosyncratic risk, and then derive it.

$$\hat{\sigma}_\varepsilon'^2 = \left(\hat{\beta}_L - \hat{\beta}_L'\right)\left(\hat{\beta}_L \hat{\sigma}_M^2\right) + \left(1 - \frac{\hat{\beta}_L'}{2}\right)\hat{\sigma}_{PL}^2 \tag{19.12a}$$

$$\hat{\sigma}_\varepsilon'^2 = \left(\hat{\beta}_H - \hat{\beta}_H'\right)\left(\hat{\beta}_H \hat{\sigma}_M^2\right) + \left(1 - \frac{\hat{\beta}_H'}{2}\right)\hat{\sigma}_{PH}^2 \tag{19.12b}$$

Setting the PRM decomposition of total risk equal to the CAPM decomposition gives us this:

$$\hat{\beta}_L^2 \hat{\sigma}_M^2 + \hat{\sigma}_{PL}^2 = \hat{\beta}_L'^2 \hat{\sigma}_M'^2 + \hat{\sigma}_\varepsilon'^2$$

$$\hat{\beta}_L^2 \hat{\sigma}_M^2 + \hat{\sigma}_{PL}^2 = \hat{\beta}_L'\left(\hat{\beta}_L' \hat{\sigma}_M'^2\right) + \hat{\sigma}_\varepsilon'^2$$

By using the formula for low CAPM $\hat{\beta}_L'$ as a ratio of covariance to market variance:

$$\hat{\beta}_L^2\hat{\sigma}_M^2 + \hat{\sigma}_{PL}^2 = \hat{\beta}_L' \frac{\sigma_{L,M}'}{\sigma_M'^2}\hat{\sigma}_M'^2 + \hat{\sigma}_\varepsilon'^2 = \hat{\beta}_L'\sigma_{L,M}' + \hat{\sigma}_\varepsilon'^2$$

Substituting the PRM decomposition of $\hat{\sigma}_{L,M}'$ gives:

$$\hat{\beta}_L^2\hat{\sigma}_M^2 + \hat{\sigma}_{PL}^2 = \hat{\beta}_L'\left(\hat{\sigma}_L\hat{\sigma}_M + \frac{1}{2}\hat{\sigma}_{PL}^2 + \frac{1}{2}\hat{\sigma}_{PL,PH}\right) + \hat{\sigma}_\varepsilon'^2$$

Moving things around a bit gives us:

$$\hat{\beta}_L^2\hat{\sigma}_M^2 + \hat{\sigma}_{PL}^2 = \hat{\beta}_L'\hat{\beta}_L\hat{\sigma}_M^2 + \frac{\hat{\beta}_L'}{2}\hat{\sigma}_{PL}^2 + \frac{\hat{\beta}_L'}{2}\hat{\sigma}_{PL,PH} + \hat{\sigma}_\varepsilon'^2$$

$$(\hat{\beta}_L - \hat{\beta}_L')\hat{\beta}_L\hat{\sigma}_M^2 + \sigma_{PL}^2 = \frac{\hat{\beta}_L'}{2}\hat{\sigma}_{PL}^2 + \frac{\hat{\beta}_L'}{2}\hat{\sigma}_{PL,PH} + \hat{\sigma}_\varepsilon'^2$$

Moving things around a bit more gives:

$$(\hat{\beta}_L - \hat{\beta}_L')\hat{\beta}_L\hat{\sigma}_M^2 = \left(\frac{\hat{\beta}_L'}{2} - 1\right)\hat{\sigma}_{PL}^2 + \frac{\hat{\beta}_L'}{2}\hat{\sigma}_{PL,PH} + \hat{\sigma}_\varepsilon'^2$$

Finally:

$$\hat{\sigma}_\varepsilon'^2 = (\hat{\beta}_L - \hat{\beta}_L')\hat{\beta}_L\hat{\sigma}_M^2 + \left(1 - \frac{\hat{\beta}_L'}{2}\right)\hat{\sigma}_{PL}^2 - \frac{\hat{\beta}_L'}{2}\hat{\sigma}_{PL,PH} \qquad (19.13a)$$

$$\hat{\sigma}_\varepsilon'^2 = (\hat{\beta}_H - \hat{\beta}_H')\hat{\beta}_H\hat{\sigma}_M^2 + \left(1 - \frac{\hat{\beta}_H'}{2}\right)\hat{\sigma}_{PH}^2 - \frac{\hat{\beta}_H'}{2}\hat{\sigma}_{PL,PH} \qquad (19.13b)$$

Assuming $\hat{\sigma}_{PL,PH} = 0$, then:

$$\hat{\sigma}_\varepsilon'^2 = (\hat{\beta}_L - \hat{\beta}_L')\hat{\beta}_L\hat{\sigma}_M^2 + \left(1 - \frac{\hat{\beta}_L'}{2}\right)\hat{\sigma}_{PL}^2 \qquad (19.12a)$$

$$\hat{\sigma}_\varepsilon'^2 = (\hat{\beta}_H - \hat{\beta}_H')\hat{\beta}_H\hat{\sigma}_M^2 + \left(1 - \frac{\hat{\beta}_H'}{2}\right)\hat{\sigma}_{PH}^2 \qquad (19.12b)$$

You can now see with precision the differences between perturbation risk and idiosyncratic risk.

For calculation purposes, the estimate of ideal market risk (19.5) should be substituted into the equation for idiosyncratic risk (19.12a and b) because it's easier to work with, resulting in equations (19.14a and b).

$$\hat{\sigma}_\varepsilon'^2 = (\hat{\beta}_L - \hat{\beta}_L')\left(\frac{\hat{\sigma}_{L,H}'}{\hat{\beta}_H}\right) + \left(1 - \frac{\hat{\beta}_L'}{2}\right)\hat{\sigma}_{PL}^2 \qquad (19.14a)$$

$$\hat{\sigma}_\varepsilon'^2 = (\hat{\beta}_H - \hat{\beta}_H')\left(\frac{\hat{\sigma}_{L,H}'}{\hat{\beta}_L}\right) + \left(1 - \frac{\hat{\beta}_H'}{2}\right)\hat{\sigma}_{PH}^2 \qquad (19.14b)$$

The ideal betas and perturbation risks are different for each entry in Table 19.1, yet each entry has the same idiosyncratic risk. Worked examples demonstrating this fact are in the next video.

Alpha Found Video 20

The Backward Perturbation Risk Model: Three Examples

Welcome back again.

This video is an attempt to interpret the numbers from Table 19.1, which were derived from example 13.1.

Table 19.1
Relationship Between Ex-Post CAPM $\widehat{\boldsymbol{\beta}}'$, Estimated Ideal $\widehat{\boldsymbol{\beta}}$s, and Estimated Perturbation Risks as Derived from Example 13.1

$Condition$	$\hat{\beta}_L$	$\hat{\beta}_H$	$\hat{\sigma}_{PL}$	$\hat{\sigma}_{PH}$	$\hat{\sigma}_{PH}/\hat{\sigma}_{PL}$	$Notes$
1	0.3889	1.6111	0.05784	0.00000	0.00000	Minimum
1	0.4000	1.6000	0.05600	0.05897	1.05306	
2	0.4060	1.5940	0.05498	0.07255	1.31966	
3	0.4150	1.5850	0.05337	0.08871	1.66207	
4	0.4229	1.5771	0.05192	0.10026	1.93118	CAPM with alpha
5	0.4350	1.5650	0.04957	0.11512	2.32264	
6	0.4645	1.5355	0.04314	0.14262	3.30606	Equal Sharpe Ratios
7	0.4693	1.5307	0.04196	0.14635	3.48771	Midpoint
8	0.4822	1.5178	0.03863	0.15550	4.02571	
9	0.5000	1.5000	0.03333	0.16667	5.00000	No intercept model
9	0.5497	1.4503	0.00000	0.19124	Undef.	Maximum

Remember that example 13.1 is made up. Right now, this is about process.

What follows is three examples from Table 19.1.

The examples are CAPM with alpha, which is condition 4; the midpoint method, which happens to be condition 7; and the zero-intercept method, which happens to be condition 9. These examples will show how the same data is subject to different interpretations. With different data, the midpoint and zero intercept methods might correspond to different conditions.

Remember CAPM has a zero-intercept expectation under OLS. PRM leads to a non-zero intercept expectation under OLS. “Zero intercept” in PRM is one of several methods of estimating ideal beta regardless of OLS expectations.

Example 20.1 (Scenario 4, CAPM with alpha)

Example 20.1 represents the volatility effect as it is commonly understood. It is CAPM through the lens of PRM.

Since scenario 4 is CAPM, it is assumed that the measurement of beta is correct. That is:

$\hat{\beta}_L' = \hat{\beta}_L = 0.422882$, and $E(\hat{\alpha}_L') = 0\%$, although realized $\hat{\alpha}_L' = 0.77118\%$.

Using (19.6a), the perturbation risk estimates are as follows:

$$\hat{\sigma}_L'^2 = \frac{\hat{\beta}_L \hat{\sigma}'_{L,H}}{\hat{\beta}_H} + \hat{\sigma}^2_{PL} = \frac{0.422882(0.0243)}{1.577117} + \hat{\sigma}^2_{PL} = .0092111$$

$$\hat{\sigma}^2_{PL} = .002695 \quad \rightarrow \quad \hat{\sigma}_{PL} = .051917$$

By similar calculation:

$$\hat{\sigma}^2_{PH} = .010052 \quad \rightarrow \quad \hat{\sigma}_{PH} = .100261$$

According to condition 4, the ratio of the perturbation risks is equal to the square root of the ratio of the ideal betas:

$$\frac{\hat{\sigma}_{PH}}{\hat{\sigma}_{PL}} = \sqrt{\frac{\hat{\beta}_H}{\hat{\beta}_L}}$$

$$\sqrt{\frac{\hat{\beta}_H}{\hat{\beta}_L}} = \sqrt{\frac{2-.422882}{.422882}} = 1.93117 = \frac{.100261}{.051917} = \frac{\hat{\sigma}_{PH}}{\hat{\sigma}_{PL}}$$

The square root of the ratio of $\hat{\beta}_H$ to $\hat{\beta}_L$ equals the ratio of perturbation risks, so the condition checks.

The estimated perturbation risks are consistent with scenario 4, with an $E(\hat{\alpha}'_L) = 0\%$, but with a measured $\hat{\alpha}'_L = 0.77118\%$. That is, if $\hat{\beta}'_L$ is correct, then the excess risk adjusted return is also correct.

The Sharpe ratios are unchanged from example 13.1 and repeated here:

$$\hat{S}'_L = 52.10\% \qquad \hat{S}'_M = 50.24\% \qquad \hat{S}'_H = 47.28\%$$

Example 20.1 shows the volatility effect as we know it.

The perturbation risks are not used in the solution process since it was assumed that the CAPM beta derived from OLS is correct. However, the perturbation risk estimates, if correct, are consistent with condition 4 and give insight into the conditions that generated the results.

Finally, idiosyncratic risk plays no role in the solution. However, idiosyncratic risk is the same for every entry in Table 19.1, and is calculated here and in the next two examples in order to demonstrate this fact.

$$\hat{\sigma}_\varepsilon'^2 = (.422882 - .422882)\frac{.0243}{(2-.422882)} + \left(\frac{1.577118}{2}\right).0026954 = 0.0021255$$

Example 20.2 (Midpoint Method)

In Example 13.1, all the parameters are known as part of an ex-ante model. It's easy to see that the parameter estimates in example 20.1 do not match example 13.1 and are therefore not correct. Some alternate method of estimating β_L and β_H must be found.

Example 20.2 uses a purely mathematical technique that does not rely on knowing or assuming an economic model. In the absence of some compelling theory, a logical estimator for β_L is the mid-point of the minimum and maximum low ideal beta estimates $(\hat{\beta}_L)$.

The formula for the midpoint estimate is derived here in two lines of algebra:

$$Mid(\hat{\beta}_L) = \frac{Min(\hat{\beta}_L) + Max(\hat{\beta}_L)}{2} = \frac{2\hat{\sigma}'_{L,H}}{2(\hat{\sigma}'_{L,H} + \hat{\sigma}'^2_H)} + \frac{2\hat{\sigma}'^2_L}{2(\hat{\sigma}'_{L,H} + \hat{\sigma}'^2_L)}$$

$$Mid(\hat{\beta}_L) = \frac{\hat{\sigma}'_{L,H}}{\hat{\sigma}'_{L,H}+\hat{\sigma}'^2_H} + \frac{\hat{\sigma}'^2_L}{\hat{\sigma}'_{L,H}+\hat{\sigma}'^2_L} \tag{20.1}$$

With this formula in hand, we calculate a midpoint estimate for the ideal beta for the low CAPM beta portfolio:

$$Mid(\hat{\beta}_L) = \frac{0.0243}{0.0243+0.10068} + \frac{0.0092111}{.0243+0.0092111} = .469302$$

Again, using equation (19.6a), the perturbation risk estimates are as follows:

$$\hat{\sigma}'^2_L = \frac{\hat{\beta}_L\hat{\sigma}'_{L,H}}{\hat{\beta}_H} + \hat{\sigma}^2_{PL} = \frac{0.46930(0.0243)}{1.53070} + \hat{\sigma}^2_{PL} = .0092111 \quad \rightarrow$$

$$\hat{\sigma}^2_{PL} = 0.001761 \quad \rightarrow \quad \hat{\sigma}_{PL} = 0.041963$$

$$\hat{\sigma}^2_{PH} = 0.021420 \quad \rightarrow \quad \hat{\sigma}_{PH} = 0.146355$$

According to condition 7:

$$\frac{\sqrt{\hat{\beta}^2_H+2\hat{\beta}_L\hat{\beta}_H}}{\hat{\beta}_L} > \frac{\hat{\sigma}_{PH}}{\hat{\sigma}_{PL}} > \frac{\hat{\beta}_H}{\hat{\beta}_L} \quad \rightarrow \quad \frac{\sqrt{1.530698^2+2(.469302)(1.530698)}}{.469302} > \frac{.146355}{.041963} > \frac{1.530698}{.469302}$$

$$\frac{\sqrt{\hat{\beta}^2_H+2\hat{\beta}_L\hat{\beta}_H}}{\hat{\beta}_L} = 4.14268 > 3.487715 > 3.26166 = \frac{\hat{\beta}_H}{\hat{\beta}_L} \quad ✓$$

The estimated perturbation risks are consistent with scenario 7, with an $E(\hat{\alpha}'_L) = 0.3070\%$, but with a measured $\hat{\alpha}'_L = 0.77118\%$. The difference, 0.4642%, is the estimated excess return after adjusting for ideal systematic risk.

The Sharpe ratios remain unchanged from example 13.1 but are repeated here:

$$\hat{S}'_L = 52.10\% \qquad \hat{S}'_M = 50.24\% \qquad \hat{S}'_H = 47.28\%$$

Example 20.2 shows that different estimation methods result in different estimates of the underlying parameters. Each estimate is consistent with the market statistics.

Because we know the original parameters found in example 13.1 are different from the estimates derived in this example, we know the midpoint method did not give the correct answer in this instance. However, the perturbation risk estimates are consistent with condition 7, and, if correct, give insight into the process that generated the results.

Finally, idiosyncratic risk plays no role in the solution.

However, as stated before, idiosyncratic risk is the same for every entry in Table 19.1 and is calculated here and in the next example in order to demonstrate this fact.

$$\hat{\sigma}_{\varepsilon}'^{2} = (.469302 - .422882)\frac{.0243}{(2-.469302)} + \left(\frac{1.577118}{2}\right).001761 = 0.0021255$$

Example 20.3 (Zero Intercept Method)

The original understanding of CAPM was the zero-intercept model. If correct, then the $\hat{\alpha}_L' = 0.77118\%$ is a statistical artifact of a misapplied method.

The way to enforce zero risk-adjusted excess return for the low and high CAPM beta portfolios is to use zero intercept regression, and set the ideal beta equal to the beta estimate generated from that process, assuming zero average perturbation return. This method will be discussed a further in a later video. For now, we will continue with the example.

Since average excess market return is 10%, and average excess return on the low CAPM beta portfolio is 5%, then $\hat{\beta}_L = 0.5$ and $\hat{\beta}_H = 1.5$.

Running through the calculations again:

$$\hat{\sigma}_L'^{2} = \frac{\hat{\beta}_L \hat{\sigma}_{L,H}'}{\hat{\beta}_H} + \hat{\sigma}_{PL}^2 = \frac{0.5(0.0243)}{1.5} + \hat{\sigma}_{PL}^2 = .0092111$$

$$\hat{\sigma}_{PL}^2 = 0.001111 \quad \rightarrow \quad \hat{\sigma}_{PL} = 0.033333$$

$$\hat{\sigma}_{PH}^2 = 0.027777 \quad \rightarrow \quad \hat{\sigma}_{PH} = 0.166666$$

According to condition 9:

$$\frac{\hat{\sigma}_{PH}}{\hat{\sigma}_{PL}} > \frac{\sqrt{\hat{\beta}_H^2 + 2\hat{\beta}_L\hat{\beta}_H}}{\hat{\beta}_L} \quad \rightarrow \quad \frac{0.166666}{.033333} > \frac{\sqrt{1.5^2+2(.5)(1.5)}}{.5} \quad \rightarrow \quad 5 > 3.873 \quad ✓$$

The estimated perturbations risks are consistent with scenario 9, with expected CAPM alpha and measured CAPM alpha equal to 0.77118%, but with zero excess risk-adjusted return as measured by PRM.

The Sharpe ratios are unchanged from example 13.1, and are repeated here:

$$\hat{S}_L' = 52.10\% \qquad \hat{S}_M' = 50.24\% \qquad \hat{S}_H' = 47.28\%$$

Example 20.3 is simply example 13.1 worked backward.

Since the standard of condition 9 is met for the PRR, the Sharpe ratios follow this pattern $\hat{S}_L' > \hat{S}_M' > \hat{S}_H'$, without there being any excess risk adjusted return as measured by ideal beta.

To complete the example, idiosyncratic risk is calculated.

$$\hat{\sigma}_{\varepsilon}'^{2} = (.500 - .422882)\frac{.0243}{(2-.500)} + \left(\frac{1.577118}{2}\right).001111 = 0.0021255$$

If we did not know that the statistics came from example 13.1, it could not be known which interpretation, Example 20.1, Example 20.2, Example 20.3, or something else, is correct. Each entry in Table 19.1 is possible along with every unlisted value between the endpoints. Each is a different interpretation of the same evidence.

The problem revealed in Table 19.1 is that the observable statistics are the same for each entry, while the unobservable statistics are different for each entry.

Observable Statistics (Same in each entry in Table 19.1)	Unobservable Statistics (Different in each entry in Table 19.1)
$\hat{\beta}'_L \;\; \hat{\beta}'_H \;\; \hat{\alpha}'_L \;\; \hat{\alpha}'_H \;\; \hat{\sigma}'_L \;\; \hat{\sigma}'_H \;\; \hat{\sigma}'_M \;\; \hat{\sigma}'_\varepsilon$	$\hat{\beta}_L \;\; \hat{\beta}_H \;\; \hat{\sigma}_L \;\; \hat{\sigma}_H \;\; \hat{\sigma}_M \;\; \hat{\sigma}_{PL} \;\; \hat{\sigma}_{PH}$

In the real world it is easy to find examples where the zero-intercept method gives an answer that is outside the range given by the minimum and maximum estimates for the low ideal beta.

Perhaps there is something missing.

Alpha Found Video 21

Another Look at the Backward Perturbation Risk Model

Welcome back again to Alpha Found.

Estimating the values of β_L and β_H has proven to be a challenge. A rigorous solution is not available.

One possible way to solve the problem of ideal beta estimates is to use the zero-intercept model for the low and high CAPM beta portfolios. Here is the reasoning: in CAPM all of the idiosyncratic risk sums to zero for the aggregate of all securities in each time period and for each security individually across time.

In the perturbation risk model this is not the case. However, the average perturbation for a well-diversified portfolio over a long period of time should average out to be close to zero since the expectation is zero and the process is mean reverting. This is just the assumptions of equation (12.4) as they were applied to (12.5a) and (12.5b). It is unlikely for an individual stock or an under-diversified portfolio to meet this standard.

These videos are about model building, not data analysis. However, in the real world it is easy to find examples where the zero-intercept model gives an answer that is outside the range given by the estimation methods shown in the last video.

There are two possibilities:

1. A large and diversified portfolio over a long period of time can generate excess risk-adjusted returns consistently for no clear reason, and therefore, the volatility effect remains a mystery to some degree.

Or

2. $\sigma_{PL,PH} \neq 0$.

Table 16.1 set out 9 scenarios and the corresponding conditions needed to meet certain risk and return expectations. The derivations of the nine conditions frequently used the phrase "assuming $\hat{\sigma}_{PL,PH} = 0$." Removing that assumption means $\sigma_{PL,PH} \neq 0$.

Table 16.1 will be called the *naïve model*. Putting $\sigma_{PL,PH} \neq 0$ back into the conditions gives us Table 21.1. Table 21.1 will be called the full model.

Table 21.1
Expected Relationship Between OLS Alphas and Sharpe Ratios
For Different Perturbation Risk Relationships with $\hat{\boldsymbol{\sigma}}_{PL,PH} \neq 0$

	Expected $\hat{\alpha}'$	**Expected $\hat{S}'$**	**Condition**	
1	$E(\hat{\alpha}'_L) < 0 < E(\hat{\alpha}'_H)$	$E(\hat{S}'_H) > \boldsymbol{E(\hat{S}'_M)} > E(\hat{S}'_L)$	$\frac{\beta_H}{\sqrt{\beta_L^2+2\beta_L\beta_H}} > \frac{\sigma_{PH}}{\sqrt{\sigma_{PL}^2+2\sigma_{PL,PH}}}$	(21.1)

2	$E(\hat{\alpha}_L') < 0 < E(\hat{\alpha}_H')$	$E(\hat{S}_H') = \boldsymbol{E(\hat{S}_M')} > E(\hat{S}_L')$	$\frac{\beta_H}{\sqrt{\beta_L^2+2\beta_L\beta_H}} = \frac{\sigma_{PH}}{\sqrt{\sigma_{PL}^2+2\sigma_{PL,PH}}}$	(21.2)
3	$E(\hat{\alpha}_L') < 0 < E(\hat{\alpha}_H')$	$\boldsymbol{E(\hat{S}_M')} > E(\hat{S}_H') > E(\hat{S}_L')$	$\sqrt{\frac{\beta_H}{\beta_L}} > \sqrt{\frac{\sigma_{PH}^2+\sigma_{PL,PH}}{\sigma_{PL}^2+\sigma_{PL,PH}}} > \frac{\beta_H}{\sqrt{\beta_L^2+2\beta_L\beta_H}}$	(21.3)
4	$E(\hat{\alpha}_L') = 0 = E(\hat{\alpha}_H')$	$\boldsymbol{E(\hat{S}_M')} > E(\hat{S}_H') > E(\hat{S}_L')$	$\sqrt{\frac{\beta_H}{\beta_L}} = \sqrt{\frac{\sigma_{PH}^2+\sigma_{PL,PH}}{\sigma_{PL}^2+\sigma_{PL,PH}}}$	(21.4)
5	$E(\hat{\alpha}_L') > 0 > E(\hat{\alpha}_H')$	$\boldsymbol{E(\hat{S}_M')} > E(\hat{S}_H') > E(\hat{S}_L')$	$\frac{\beta_H}{\beta_L} > \sqrt{\frac{\sigma_{PH}^2+\sigma_{PL,PH}}{\sigma_{PL}^2+\sigma_{PL,PH}}} > \sqrt{\frac{\beta_H}{\beta_L}}$	(21.5)
6	$E(\hat{\alpha}_L') > 0 > E(\hat{\alpha}_H')$	$\boldsymbol{E(\hat{S}_M')} > E(\hat{S}_H') = E(\hat{S}_L')$	$\frac{\sqrt{\beta_H^2+2\beta_L\beta_H}}{\beta_L} > \frac{\sigma_{PH}}{\sigma_{PH}} = \frac{\beta_H}{\beta_L}$	(21.6)
7	$E(\hat{\alpha}_L') > 0 > E(\hat{\alpha}_H')$	$\boldsymbol{E(\hat{S}_M')} > E(\hat{S}_L') > E(\hat{S}_H')$	$\frac{\sqrt{\beta_H^2+2\beta_L\beta_H}}{\beta_L} > \frac{\sqrt{\sigma_{PH}^2+2\sigma_{PL,PH}}}{\hat{\sigma}_{PL}} > \frac{\beta_H}{\beta_L}$	(21.7)
8	$E(\hat{\alpha}_L') > 0 > E(\hat{\alpha}_H')$	$\boldsymbol{E(\hat{S}_M')} = E(\hat{S}_L') > E(\hat{S}_H')$	$\frac{\sqrt{\beta_H^2+2\beta_L\beta_H}}{\beta_L} = \frac{\sqrt{\sigma_{PH}^2+2\sigma_{PL,PH}}}{\sigma_{PL}} > \frac{\beta_H}{\beta_L}$	(21.8)
9	$E(\hat{\alpha}_L') > 0 > E(\hat{\alpha}_H')$	$E(\hat{S}_L') > \boldsymbol{E(\hat{S}_M')} > E(\hat{S}_H')$	$\frac{\sqrt{\sigma_{PH}^2+2\sigma_{PL,PH}}}{\sigma_{PL}} > \frac{\sqrt{\beta_H^2+2\beta_L\beta_H}}{\beta_L}$	(21.9)

The perturbation risk ratio has a fluid definition in the full model.

Conditions 1, 2 and 3 have this definition for the perturbation risk ratio:

Conditions 1, 2, 3 $\qquad$ $\text{Full PRR} = \frac{\sigma_{PH}}{\sqrt{\sigma_{PL}^2+2\sigma_{PL,PH}}}$

Conditions 4 and 5 have a different definition:

Conditions 4, 5 $\qquad$ $\text{Full PRR} = \sqrt{\frac{\sigma_{PH}^2+\sigma_{PL,PH}}{\sigma_{PL}^2+\sigma_{PL,PH}}}$

Condition 6 has the same definition as in the naïve model in Table 16.1:

Condition 6 $\qquad$ $\text{Full PRR} = \frac{\sigma_{PH}}{\sigma_{PL}}$

Finally, conditions 7, 8, and 9 have this definition:

Conditions 7, 8, 9 $\qquad$ $\text{Full PRR} = \frac{\sqrt{\sigma_{PH}^2+2\sigma_{PL,PH}}}{\sigma_{PL}}$

The derivations of the conditions in Table 21.1 result in some apparent inconsistencies.

For instance, the derivation of condition 3 in Table 21.1 leads to the following:

$$\sqrt{\frac{\beta_H}{\beta_L}} > \sqrt{\frac{\sigma_{PH}^2+\sigma_{PL,PH}}{\sigma_{PL}^2+\sigma_{PL,PH}}} > \sqrt{\frac{\sigma_{PH}^2}{\sigma_{PL}^2+2\sigma_{PL,PH}}} > \frac{\beta_H}{\sqrt{\beta_L^2+2\beta_L\beta_H}}$$

There is no longer one middle term but two. Either of the two middle terms is sufficient, and neither contradicts the other. Also, the end point conditions are the important conditions, and there is no difficulty with them.

The form given for each condition in Table 21.1 seems best for the goals of these videos. Others may prefer a different form.

It is important to notice something about the conditions of Table 21.1 that is not in Table 16.1.

β_H corresponds with σ_{PH}, β_L corresponds with σ_{PL}, and $\beta_L\beta_H$ corresponds with $\sigma_{PL,PH} = \rho_{PL,PH}\sigma_{PL}\sigma_{PH}$.

The corresponding pieces do not have the same value, but have they often have the same place in the structure of the conditions.

For example, look at condition 7:

$$\frac{\sqrt{\beta_H^2+2\beta_L\beta_H}}{\beta_L} > \frac{\sqrt{\sigma_{PH}^2+2\sigma_{PL,PH}}}{\sigma_{PL}} > \frac{\beta_H}{\beta_L} \tag{21.7}$$

The two left-hand terms have the same structure.

Including the covariance between perturbations ($\sigma_{PL,PH} \neq 0$) means all the previously derived formulas need to be rewritten, and set to music.

(By rearranging equation (12.19) and substituting into (12.11a) and (12.11b)):

$$\sigma_L'^2 = \frac{\beta_L}{\beta_H}\left(\sigma_{L,H}' - \sigma_{PL,PH}\right) + \sigma_{PL}^2 \tag{21.10a}$$

$$\sigma_H'^2 = \frac{\beta_H}{\beta_L}\left(\sigma_{L,H}' - \sigma_{PL,PH}\right) + \sigma_{PH}^2 \tag{21.10b}$$

From (12.14)
$$\sigma_M'^2 = \sigma_M^2 + \frac{\sigma_{PL}^2}{4} + \frac{\sigma_{PH}^2}{4} + \frac{\sigma_{PL,PH}}{2} \tag{21.11}$$

From (12.17a)
$$\sigma_{L,M}' = \beta_L\sigma_M^2 + \frac{\sigma_{PL}^2}{2} + \frac{\sigma_{PL,PH}}{2} \tag{21.12a}$$

From (12.17b)
$$\sigma_{H,M}' = \beta_H\sigma_M^2 + \frac{\sigma_{PH}^2}{2} + \frac{\sigma_{PL,PH}}{2} \tag{21.12b}$$

From (12.19)
$$\sigma_{L,H}' = \beta_L\beta_H\sigma_M^2 + \sigma_{PL,PH} \tag{21.13}$$

From (12.21a)
$$\beta_L' = \frac{\beta_L\sigma_M^2+\frac{\sigma_{PL}^2}{2}+\frac{\sigma_{PL,PH}}{2}}{\sigma_M'^2} \tag{21.14a}$$

From (12.21b) $$\beta_H' = \frac{\beta_H\sigma_M^2 + \frac{\sigma_{PH}^2}{2} + \frac{\sigma_{PL,PH}}{2}}{\sigma_M'^2} \tag{21.14b}$$

Rearranging (12.19) $$\sigma_M^2 = \frac{\sigma_{L,H}' - \sigma_{PL,PH}}{\beta_L\beta_H} \tag{21.15}$$

From (19.13a) $$\sigma_\varepsilon'^2 = (\beta_L - \beta_L')\beta_L\sigma_M^2 + \left(\frac{\beta_H'}{2}\right)\sigma_{PL}^2 - \frac{\beta_L'}{2}\sigma_{PL,PH} \tag{21.16a}$$

From (19.13b) $$\sigma_\varepsilon'^2 = (\beta_H - \beta_H')\beta_H\sigma_M^2 + \left(\frac{\beta_L'}{2}\right)\sigma_{PH}^2 - \frac{\beta_H'}{2}\sigma_{PL,PH} \tag{21.16b}$$

There are several hopeful avenues of approach available to solve the equations above. Unfortunately, they don't work. What is left are approximation methods.

Full condition 9 is repeated here:

$$\frac{\sqrt{\sigma_{PH}^2 + 2\sigma_{PL,PH}}}{\sigma_{PL}} > \frac{\sqrt{\beta_H^2 + 2\beta_L\beta_H}}{\beta_L}$$

Two substitutions into full condition 9 are helpful. Equations (21.10a) and (21.10b) can be rearranged to give:

$$\sigma_{PL}^2 = \sigma_L'^2 - \frac{\beta_L}{\beta_H}\left(\sigma_{L,H}' - \sigma_{PL,PH}\right) \tag{21.17a}$$

$$\sigma_{PH}^2 = \sigma_H'^2 - \frac{\beta_H}{\beta_L}\left(\sigma_{L,H}' - \sigma_{PL,PH}\right) \tag{21.17b}$$

The two substitutions have this result:

$$\sqrt{\frac{\sigma_H'^2 - \frac{\beta_H}{\beta_L}(\sigma_{L,H}' - \sigma_{PL,PH}) + 2\sigma_{PL,PH}}{\hat{\sigma}_L'^2 - \frac{\beta_L}{\beta_H}(\sigma_{L,H}' - \sigma_{PL,PH})}} > \frac{\sqrt{\beta_H^2 + 2\beta_L\beta_H}}{\beta_L} \tag{21.18}$$

In this mess of an inequality, all the pieces are known except $\sigma_{PL,PH}$.

To ensure that the denominator and ideal market variance of equation (21.18) are positive, two more conditions must hold:

One, the denominator must be greater than zero:

1. $\sigma_L'^2 - \frac{\beta_L}{\beta_H}\left(\sigma_{L,H}' - \sigma_{PL,PH}\right) > 0$ (21.19)

And two, the covariance between the high and low CAPM beta portfolios must be greater than the covariance between the perturbations:

2. $\sigma_{L,H}' < \sigma_{PL,PH}$ (21.20)

Condition 2 is necessarily true but I will skip the proof.

As a reminder:

$$\sigma_{PL,PH} = \rho_{PL,PH}\sigma_{PL}\sigma_{PH}$$

To examine inequality (21.18), quarterly total return data for the components of the S&P 500 from the first quarter of 2015 through the fourth quarter of 2019 were used. Only those companies that were part of the S&P 500 for the entire period were included. Individual stocks were rebalanced to their initial weights each quarter**.**

Without this procedure the market statistics are not as clean as is needed to make the assumptions work. Stated another way, the S&P 500 is really a different index from one quarter to the next, unless sameness is imposed.

It would be nice to compare to the same index in each time period, not a moving target.

Let me squeeze in the disclaimer that this data set, as constructed, suffers from survivorship bias and other problems.

The following statistics were gathered while using the zero-intercept estimate for β_L and β_H. I'm not going to read through them, but here they are for reference.

$$\hat{\sigma}_M'^2 = 0.003067 \qquad \hat{\sigma}_{L,H}' = 0.002288$$

$$\hat{\sigma}_L'^2 = 0.001256 \qquad \hat{\sigma}_H'^2 = 0.006437$$

$$\hat{\beta}_L' = 0.577742 \qquad \hat{\beta}_H' = 1.422259$$

$$\hat{\beta}_L = 0.943391 \qquad \hat{\beta}_H = 1.056609$$

$$\hat{S}_L' = 0.706524 \qquad \hat{S}_M' = 0.479300 \qquad \hat{S}_H' = 0.349587$$

$$\hat{\rho}_{L,H}' = 0.80453 \qquad \hat{\rho}_{L,M}' = 0.90274 \qquad \hat{\rho}_{H,M}' = 0.98177$$

Since $\hat{S}_L' > \hat{S}_M' > \hat{S}_H'$, the full condition of scenario 9 with the substitutions of equations (21.17a) and (21.17b) will be examined.

$$\sqrt{\frac{\hat{\sigma}_H'^2 - \frac{\hat{\beta}_H}{\hat{\beta}_L}(\hat{\sigma}_{L,H}' - \hat{\sigma}_{PL,PH}) + 2\hat{\sigma}_{PL,PH}}{\hat{\sigma}_L'^2 - \frac{\hat{\beta}_L}{\hat{\beta}_H}(\hat{\sigma}_{L,H}' - \hat{\sigma}_{PL,PH})}} > \frac{\sqrt{\hat{\beta}_H^2 + 2\hat{\beta}_L\hat{\beta}_H}}{\hat{\beta}_L} = 1.869$$

Making a few calculations in the background:

$\hat{\sigma}_L'^2 - \frac{\hat{\beta}_L}{\hat{\beta}_H}\left(\hat{\sigma}_{L,H}' - \sigma_{PL,PH}\right) > 0$ when $\hat{\sigma}_{PL,PH} > 0.0008807$.

$\hat{\sigma}_{L,H}' > \hat{\sigma}_{PL,PH}$ when $\hat{\sigma}_{PL,PH} < 0.0022877$.

Therefore:

$$0.0008807 < \hat{\sigma}_{PL,PH} < 0.0022877$$

Once $\hat{\sigma}_{PL,PH}$ has been estimated, $\hat{\sigma}^2_{PL}$, $\hat{\sigma}^2_{PH}$, and $\hat{\sigma}^2_M$ can be calculated from equations (21.17a), (21.17b), and (21.15).

Making a table of potential values of $\hat{\sigma}_{PL,PH}$ leads to another constraint. The lower part of the range for $\hat{\sigma}_{PL,PH}$ leads to correlation above 1 ($\hat{\rho}_{PL,PH} > 1$). Consequently, the range is constrained further such that $\hat{\sigma}_{PL,PH}$ is between the numbers shown.

$$0.00118156 < \hat{\sigma}_{PL,PH} < 0.0022877$$

Table 21.2 is the result.

Table 21.2

Estimated Ex-Post Covariance and Variance of Perturbations and Full *PRR* for Realized Market Data January 2015 – December 2019

	$\hat{\sigma}_{PL,PH}$	$\hat{\sigma}^2_{PL}$	$\hat{\sigma}^2_{PH}$	$\hat{\rho}_{PL,PH}$	$\hat{\sigma}^2_M$	$\hat{\sigma}_M$	PRR
1	0.00118	0.00027	0.00520	1.00000	0.00111	3.33%	5.31
2	0.00129	0.00037	0.00532	0.92417	0.00100	3.16%	4.64
3	0.00140	0.00047	0.00545	0.88049	0.00089	2.98%	4.21
4	0.00151	0.00056	0.00557	0.85324	0.00078	2.79%	3.90
5	0.00162	0.00066	0.00569	0.83549	0.00067	2.58%	3.67
6	0.00173	0.00076	0.00582	0.82368	0.00055	2.36%	3.49
7	0.00185	0.00086	0.00594	0.81579	0.00044	2.11%	3.34
8	0.00196	0.00096	0.00606	0.81059	0.00033	1.82%	3.22
9	0.00207	0.00106	0.00619	0.80731	0.00022	1.49%	3.12
10	0.00218	0.00116	0.00631	0.80541	0.00011	1.05%	3.04
11	0.00229	0.00126	0.00644	0.80452	0.00000	0.00%	2.96

The values of the statistics found in each row are all consistent with the market statistics from January 2015 – December 2019.

Table 21.2 contains a strong relationship for the correlations, $0.8045 < \hat{\rho}_{PL,PH} < 1.00$.

This high correlation between the perturbations is the reason the market perturbations can be so large. If the perturbations were independent, it might be expected that the average across all securities would approach zero in each time period.

In the current state of model development, there is no quantitative reason to prefer one set of values over another in Table 21.2. Subjectively, a smaller value for $\sigma_{PL,PH}$ seems likely, but this requires $\rho_{PL,PH}$ to be close to one, which seems unlikely.

The range of values in Table 21.2 for estimated quarterly ideal market and perturbation standard deviations are as follows:

$$0.00\% < \hat{\sigma}_M < 3.33\%$$

$$1.64\% < \hat{\sigma}_{PL} < 3.55\%$$

$$7.21\% < \hat{\sigma}_{PH} < 8.02\%$$

Curiously, the range for $\hat{\sigma}_{PL}$ is greater than the range for $\hat{\sigma}_{PH}$.

Choosing a value for perturbation covariance in the middle of the range of Table 21.2 will attribute about one-fourth of the market variance to ideal market variance. The rest comes from perturbation risk. The time period chosen was one of low market volatility. This may provide insight into the problem of quantifying excess volatility.

Stated differently, excess volatility by itself is logically the cause of most or all of the volatility effect because of its effect on OLS regression calculations. (I'm going with most.)

Though only about a fourth of the variance is coming from the ideal market risk, nonetheless, every entry in Table 21.2 leads to the correct values for the correlation statistics below.

$$\hat{\rho}'_{L,H} = 0.80453$$

$$\hat{\rho}'_{L,M} = 0.90274$$

$$\hat{\rho}'_{H,M} = 0.98177$$

Lastly, though the correlation between perturbations is different in each entry, each entry still generates the same idiosyncratic risk.

More details are in the next video.

Alpha Found Video 22

Details and Generalizations Part 1

Welcome back again to part 22.

This video is first of four that will provide more detail on the backward perturbation risk model. That is, there will be less computing and more modeling.

The mathematical notation and ideas are a step up. There is a lot on the screen that will be not be verbalized, though if you want to understand everything, stopping the video occasionally is your best method, or, if you are more ambitious, you could download the video transcript from the series website, and study the transcript.

One goal is to define perturbation alpha. Perturbation alpha will play the same structural role as CAPM alpha does in the equation for CAPM, but its interpretation is quite different. For instance, in CAPM, alpha is a constant that is estimated by OLS. In PRM, perturbation alpha is a random variable. More on that in video 25.

Previously it was noted that expected Sharpe ratios for the low and high CAPM beta portfolios are equal under full condition 6.

Let's assume a computational convenience. The expected low perturbation in period *t* given the market perturbation in period *t* equals the low ideal beta times the market perturbation:

$$E(R_{PLt}|R_{PMt}) = \beta_L R_{PMt} \tag{22.1}$$

It will turn out that this is not a correct conditional expectation, but it's close and will help for a while.

With the conditional expectation of equation (22.1) we have:

$$E(R'_{Lt}|R_{PMt}) - r_f = \beta_L\left(R'_{Mt} - r_f\right)$$

The expected total return of the low CAPM beta portfolio in period *t* given the market perturbation minus the risk-free rate equals the low ideal beta times the excess return of the market.

You can follow the algebra:

$$= \beta_L\left(R_{Mt} + R_{PMt} - r_f\right)$$

$$= \beta_L\left(R_{Mt} - r_f\right) + \beta_L(R_{PMt})$$

With the result that:

$$E(R'_{Lt}|R_{PMt}) - r_f = \beta_L\left(R_{Mt} - r_f\right) + E(R_{PLt}) \tag{22.2}$$

Where:

$$E(R_{PLt}|R_{PMt}) = \beta_L R_{PMt} \tag{22.1}$$

Like most expectations, it's not usually realized. If the degree to which it does not hold in a given time period is called h, then the low perturbation in period t equals the sum of the low ideal beta plus the low h

multiplied by the market perturbation.

$$R_{PLt} = (\beta_L + h_{Lt})R_{PMt} \tag{22.3a}$$

$$R_{PHt} = (\beta_H + h_{Ht})R_{PMt} \tag{22.3b}$$

The average low perturbation equals the low ideal beta times the average market perturbation plus the average of the perturbations not explained by the market perturbations.

$$\frac{1}{T}\sum R_{PLt} = \frac{1}{T}\sum(\beta_L + h_{Lt})R_{PMt} \quad \rightarrow \quad \bar{R}_{PL} = \frac{1}{T}\sum \beta_L R_{PMt} + \frac{1}{T}\sum h_{Lt} R_{PMt}$$

$$\bar{R}_{PL} = \beta_L \bar{R}_{PM} + \frac{1}{T}\sum h_{Lt} R_{PMt} \tag{22.4a}$$

$$\bar{R}_{PH} = \beta_H \bar{R}_{PM} + \frac{1}{T}\sum h_{Ht} R_{PMt} \tag{22.4b}$$

I think this is one case where the math is easier to understand than the English describing the math.

The *h*s add up to zero in each time period and as an average over time:

$$h_L = -h_H \tag{22.5}$$

Such that the weighted sum of the individual perturbations in each time period when summed and averaged across time equals the average perturbation.

$$\frac{1}{T}\sum\sum w_i(\beta_i + h_{it})R_{PMt} = \bar{R}_{PM} \tag{22.6}$$

Where the sum of the weights is 1, and the weighted sum of the ideal betas is 1, and the weighted sum of the *h*s is zero in each time period and are therefore zero across time, but not necessarily zero for any given security or portfolio.

$$\sum w_i = 1 \tag{22.7}$$

$$\sum w_i(\beta_i) = 1 \tag{22.8}$$

$$\sum\sum w_i(h_{it}) = 0 \tag{22.9}$$

Remember that the β_i is held constant, h_i is a variable, and R_{PM} is a variable. Therefore, we define perturbation alpha in this manner: the low perturbation alpha equals the low *h* for a given time period times the market perturbation for that time period.

$$\alpha_{PLt} = h_{Lt}R_{PMt} \tag{22.10a}$$

$$\alpha_{PHt} = h_{Ht}R_{PMt} \tag{22.10b}$$

The formula for average perturbation alpha follows immediately:

$$\bar{\alpha}_{PL} = \frac{1}{T}\sum h_{Lt} R_{PMt} \tag{22.11a}$$

$$\bar{\alpha}_{PH} = \frac{1}{T}\sum h_{Ht} R_{PMt} \tag{22.11b}$$

$$\bar{\alpha}_{PL} = -\bar{\alpha}_{PH} \tag{22.12}$$

$$\sum w_i(\bar{\alpha}_{Pi}) = 0 \tag{22.13}$$

Where once again the weighted average alpha is zero.

Note 1. The interpretation of perturbation alpha is more subtle in the PRM than the interpretation of alpha in CAPM. If the market perturbation is positive in a given time period, then a positive *h* means greater return from the perturbation. If the market perturbation is negative in a given time period, then a positive *h* means lesser return from the perturbation.

If *h* is typically positive for a given asset or portfolio, over many time periods, it means higher standard deviation without higher arithmetic average return.

Note 2. The perturbations might or might not sum to zero in any time period or across time periods, but the perturbation alphas necessarily sum to zero in each time period.

Note 3. It is likely that $\bar{R}_{PM} \approx 0$ over time, while $\bar{\alpha}_{PL} = -\bar{\alpha}_{PH} \neq 0$. However, the modeling to this point has been largely based on $\bar{R}_{PM} = 0$, and $\bar{\alpha}_{PL} = -\bar{\alpha}_{PH} = 0$. However, it is likely $\bar{\alpha}_{PL} = -\bar{\alpha}_{PH} \approx 0$.

Note 4. It is the perturbations themselves that are the excess risk adjusted returns, not the alphas. Although the alphas are part of the perturbations.

Note 5. In the PRM it is possible that all portfolios and the market itself will have excess risk adjusted return or deficient risk-adjusted return. That is, the market might be considered too high or too low.

Note 6. By setting $\rho_{PL,PH} = 1$, and by setting the individual perturbations equal to the ideal beta times the market perturbation, CAPM beta will equal ideal beta, CAPM alphas will equal zero, all Sharpe ratios will be equal, and there will be zero idiosyncratic risk.

I should prove this.

Proof of note 6.

Here is the PRM decomposition of the low CAPM beta with the covariance term expanded.

$$\beta_L' = \frac{\beta_L\sigma_M^2 + \frac{\sigma_{PL}^2}{2} + \frac{\sigma_{PL,PH}}{2}}{\sigma_M'^2} = \frac{\beta_L\sigma_M^2 + \frac{\sigma_{PL}^2}{2} + \frac{\rho_{PL,PH}\sigma_{PL}\sigma_{PH}}{2}}{\hat{\sigma}_M'^2} \tag{12.21a}$$

If $\rho_{PL,PH} = 1$, then h_L is a constant, and the correlation term drops out of the PRM decomposition.

$$\beta_L' = \frac{\beta_L\sigma_M^2 + \frac{\sigma_{PL}^2}{2} + \frac{\sigma_{PL}\sigma_{PH}}{2}}{\sigma_M'^2}$$

Since $\rho_{PL,PH} = 1$, and by assumption the individual perturbation risks are equal to the ideal beta times the market perturbation risk (which if h_L is a constant then $h_{Lt} = 0$ for all *t*), then substituting the individual ideal betas times the market perturbation risk for individual perturbation risks:

$$\beta_L\sigma_{PM} = \sigma_{PL}, \qquad \beta_H\sigma_{PM} = \sigma_{PH} \tag{22.14}$$

$$\beta_L' = \frac{\beta_L\sigma_M^2 + \frac{\sigma_{PL}^2}{2} + \frac{\sigma_{PL}\sigma_{PH}}{2}}{\sigma_M'^2} = \frac{\beta_L\sigma_M^2 + \frac{\beta_L^2\sigma_{PM}^2}{2} + \frac{\beta_L\beta_H\sigma_{PM}^2}{2}}{\sigma_M'^2}$$

and factoring:

$$\beta_L' = \frac{\beta_L\sigma_M^2 + \beta_L\left(\frac{\beta_L\sigma_{PM}^2}{2} + \frac{\beta_H\sigma_{PM}^2}{2}\right)}{\sigma_M'^2}$$

and factoring again:

$$\beta_L' = \frac{\beta_L\sigma_M^2 + \beta_L\left(\frac{(\beta_L + \beta_H)\sigma_{PM}^2}{2}\right)}{\sigma_M'^2}$$

Simplifying:

$$\beta_L' = \frac{\beta_L\sigma_M^2 + \beta_L\left(\frac{(2)\sigma_{PM}^2}{2}\right)}{\sigma_M'^2}$$

Canceling:

$$\beta_L' = \frac{\beta_L\sigma_M^2 + \beta_L\sigma_M^2}{\hat{\sigma}_M'^2}$$

factoring yet again, and canceling:

$$\beta_L' = \frac{\beta_L(\sigma_M^2 + \sigma_M^2)}{(\sigma_M^2 + \sigma_M^2)} = \beta_L$$

Therefore,

$$\beta_L' = \beta_L$$

given $\rho_{PL,PH} = 1,\ \beta_L\sigma_{PM} = \sigma_{PL}$, and $\beta_H\sigma_{PM} = \sigma_{PH}$.

Furthermore, since $\beta_L' = \beta_L$ then the expected CAPM alphas are zero. If the expected CAPM alphas are zero, then full condition 4 should be met. Since the Sharpe ratios are equal, full condition 6 should also be met.

Superficially full condition 4 is at odds with full condition 6.

Full condition 4 is:

$$\sqrt{\frac{\beta_H}{\beta_L}} = \sqrt{\frac{\sigma_{PH}^2 + \sigma_{PL,PH}}{\sigma_{PL}^2 + \sigma_{PL,PH}}} \tag{21.4}$$

Equivalently, by squaring both sides:

$$\frac{\beta_H}{\beta_L} = \frac{\sigma_{PH}^2 + \sigma_{PL,PH}}{\sigma_{PL}^2 + \sigma_{PL,PH}}$$

And full condition 6 is:

$$\frac{\beta_H}{\beta_L} = \frac{\sigma_{PH}}{\sigma_{PL}} \tag{21.6}$$

For both to be correct at the same time, the full PRR of condition 6 should be equal to the full PRR of condition 4.

By setting them equal, and expanding the covariance term:

$$\frac{\sigma_{PH}}{\sigma_{PL}} = \frac{\sigma_{PH}^2 + \sigma_{PL,PH}}{\sigma_{PL}^2 + \sigma_{PL,PH}} = \frac{\sigma_{PH}^2 + \rho_{PL,PH}\sigma_{PL}\sigma_{PH}}{\sigma_{PL}^2 + \rho_{PL,PH}\sigma_{PL}\sigma_{PH}}$$

Then since the correlation between perturbations is 1 and drops out, and again the individual perturbation risks equal the ideal beta times the market perturbation risk:

$$\frac{\sigma_{PH}}{\sigma_{PL}} = \frac{\sigma_{PH}^2 + \rho_{PL,PH}\sigma_{PL}\sigma_{PH}}{\sigma_{PL}^2 + \rho_{PL,PH}\sigma_{PL}\sigma_{PH}} \quad \rightarrow$$

$$\frac{\beta_H\sigma_{PM}}{\beta_L\sigma_{PM}} = \frac{\beta_H^2\sigma_{PM}^2 + \beta_L\beta_H\sigma_{PM}\sigma_{PM}}{\beta_L^2\hat{\sigma}_{PM}^2 + \beta_L\beta_H\sigma_{PM}\sigma_{PM}}$$

All of the σ_{PM} terms cancel leaving:

$$\frac{\beta_H}{\beta_L} = \frac{\beta_H^2 + \beta_L\beta_H}{\beta_L^2 + \beta_L\beta_H} = \frac{\beta_H(\beta_H + \beta_L)}{\beta_L(\beta_H + \beta_L)} = \frac{\beta_H}{\beta_L}$$

Therefore, when these assumptions hold:

$$\rho_{PL,PH} = 1, \qquad R_{PLt} = \beta_L R_{PMt}, \qquad R_{PHt} = \beta_H R_{PMt}$$

The CAPM beta will equal the ideal beta, expected CAPM alphas will be zero, Sharpe ratios will be equal, and there will be zero idiosyncratic risk.

Alpha Found Video 23

Details and Generalizations Part 2

Welcome back again to Alpha Found part 23.

The goal of this video is to derive the following two correct equations for the parameterized portfolio perturbation risk:

$$\sigma_{PL} = \sigma_{PM}\sqrt{\left(\beta_L + E(h_L)\right)^2 + \sigma_{h_L}^2} \tag{23.1a}$$

$$\sigma_{PH} = \sigma_{PM}\sqrt{\left(\beta_H + E(h_H)\right)^2 + \sigma_{h_H}^2} \tag{23.1b}$$

And then see what happens when they are substituted into the 9 full conditions.

Beginning with the low perturbation return in period *t* equal to the low ideal beta plus the low *h* in period *t* times the market perturbation in period *t*:

$$R_{PLt} = (\beta_L + h_{Lt})R_{PMt} \tag{22.3a}$$

Take the variance of both sides:

$$var(R_{PL}) = var\left((\beta_L + h_L)R_{PM}\right)$$

The right-hand side is the variance of the product of two independent random variables which means:

$$var(R_{PL}) = var(\beta_L + h_L)var(R_{PM}) + var(\beta_L + h_L)\left(E(R_{PM})\right)^2 + \left(E(\beta_L + h_L)\right)^2 var(R_{PM})$$

The expectation of the market perturbations is zero so the red term in the middle drops out.

$$var(R_{PL}) = var(h_L)var(R_{PM}) + 0 + \left(E(\beta_L + h_L)\right)^2 var(R_{PM})$$

With a change to a more pleasant notation:

$$\sigma_{PL}^2 = \sigma_{h_L}^2\sigma_{PM}^2 + \left(\beta_L + E(h_L)\right)^2\sigma_{PM}^2$$

which can be rearranged to:

$$\sigma_{PL}^2 = \sigma_{PM}^2\left(\left(\beta_L + E(h_L)\right)^2 + \sigma_{h_L}^2\right) \tag{23.2a}$$

$$\sigma_{PH}^2 = \sigma_{PM}^2\left(\left(\beta_H + E(h_H)\right)^2 + \sigma_{h_H}^2\right) \tag{23.2b}$$

Where necessarily the variance of the low *h* equals the variance of the high *h*:

$$\sigma_{h_L}^2 = \sigma_{h_H}^2 \tag{23.3}$$

And the expectation of the low h equals minus the expectation of the high h:

$$E(h_L) = -E(h_H) \tag{22.5}$$

Taking the square root of both sides of equation (23.2a):

$$\sigma_{PL} = \sigma_{PM}\sqrt{\left(\beta_L + E(h_L)\right)^2 + \sigma_{h_L}^2} \tag{23.1a}$$

Full condition 6 (21.6) expands and cancels as follows:

$$\frac{\beta_H}{\beta_L} = \frac{\sigma_{PH}}{\sigma_{LH}} \quad \rightarrow \quad \frac{\beta_H}{\beta_L} = \frac{\sigma_{PM}\sqrt{\left(\beta_H+E(h_H)\right)^2+\sigma_{h_H}^2}}{\sigma_{PM}\sqrt{\left(\beta_L+E(h_L)\right)^2+\sigma_{h_L}^2}} = \frac{\sqrt{\left(\beta_H+E(h_H)\right)^2+\sigma_{h_H}^2}}{\sqrt{\left(\beta_L+E(h_L)\right)^2+\sigma_{h_L}^2}}$$

$$\frac{\beta_H}{\beta_L} = \frac{\sqrt{\left(\beta_H+E(h_H)\right)^2+\sigma_{h_H}^2}}{\sqrt{\left(\beta_L+E(h_L)\right)^2+\sigma_{h_L}^2}} \tag{23.4}$$

The market perturbation risk drops out all together from the ratio.

Squaring both sides of equation (23.4) gives:

$$\frac{\beta_H^2}{\beta_L^2} = \frac{\left(\beta_H+E(h_H)\right)^2+\sigma_{h_H}^2}{\left(\beta_L+E(h_L)\right)^2+\sigma_{h_L}^2} \tag{23.5}$$

Condition 6 is the condition necessary for the equality of Sharpe ratios.

Equality of Sharpe ratios is natural to capital market theory in that average excess return should be proportional to risk.

Here lies a major insight: For equation (23.5) to hold true it is necessary for:

$$E(h_H) > 0 > E(h_L) \tag{23.6}$$

This can be confirmed numerically if, after establishing the values for the ideal betas, we let $E(h_H) = 0 = E(h_L)$ along with giving any positive value for $\sigma_{h_L}^2 = \sigma_{h_H}^2$. The ratio will be less than it needs to be for the equality of Sharpe ratios. Only if $E(h_H) > 0 > E(h_L)$ can the ratio work out.

If equations (23.2a) and (23.2b) are substituted into full condition 9 (21.9), and the market perturbation risk is canceled out, a new expression results:

$$\frac{\sqrt{\left(\left(\beta_H+E(h_H)\right)^2+\sigma_{h_H}^2\right)+2\rho_{PL,PH}\sqrt{\left(\left(\beta_H+E(h_H)\right)^2+\sigma_{h_H}^2\right)\left(\left(\beta_L+E(h_L)\right)^2+\sigma_{h_L}^2\right)}}}{\sqrt{\left(\beta_L+E(h_L)\right)^2+\sigma_{h_L}^2}} > \frac{\sqrt{\beta_H^2+2\beta_L\beta_H}}{\beta_L} \tag{23.7}$$

What a mess. Here is the proof:

$$\frac{\sqrt{\sigma_{PH}^2+2\sigma_{PL,PH}}}{\sigma_{PL}} > \frac{\sqrt{\beta_H^2+2\beta_L\beta_H}}{\beta_L} \quad \rightarrow \quad \frac{\sqrt{\sigma_{PH}^2+2\rho_{PL,PH}\sigma_{PL}\sigma_{PH}}}{\sigma_{PL}} > \frac{\sqrt{\beta_H^2+2\beta_L\beta_H}}{\beta_L}$$

Making this substitution:

$$\sigma_{PL}^2 = \sigma_{PM}^2\left(\left(\beta_L + E(h_L)\right)^2 + \sigma_{h_L}^2\right) \tag{23.2a}$$

$$\sigma_{PH}^2 = \sigma_{PM}^2\left(\left(\beta_H + E(h_H)\right)^2 + \sigma_{h_H}^2\right) \tag{23.2b}$$

$$\sqrt{\frac{\left(\left(\beta_H+E(h_H)\right)^2+\sigma_{h_H}^2\right)\sigma_{PM}^2+2\rho_{PL,PH}\sigma_{PM}^2\sqrt{\left(\left(\beta_H+E(h_H)\right)^2+\sigma_{h_H}^2\right)\left(\left(\beta_L+E(h_L)\right)^2+\sigma_{h_L}^2\right)}}{\left(\left(\beta_L+E(h_L)\right)^2+\sigma_{h_L}^2\right)\sigma_{PM}^2}} > \frac{\sqrt{\beta_H^2+2\beta_L\beta_H}}{\beta_L}$$

The market perturbation risk cancels leaving:

$$\sqrt{\frac{\left(\left(\beta_H+E(h_H)\right)^2+\sigma_{h_H}^2\right)+2\rho_{PL,PH}\sqrt{\left(\left(\beta_H+E(h_H)\right)^2+\sigma_{h_H}^2\right)\left(\left(\beta_L+E(h_L)\right)^2+\sigma_{h_L}^2\right)}}{\left(\left(\beta_L+E(h_L)\right)^2+\sigma_{h_L}^2\right)}} > \frac{\sqrt{\beta_H^2+2\beta_L\beta_H}}{\beta_L} \tag{23.7}$$

Squaring both sides:

$$\frac{\left(\left(\beta_H+E(h_H)\right)^2+\sigma_{h_H}^2\right)+2\rho_{PL,PH}\sqrt{\left(\left(\beta_H+E(h_H)\right)^2+\sigma_{h_H}^2\right)\left(\left(\beta_L+E(h_L)\right)^2+\sigma_{h_L}^2\right)}}{\left(\left(\beta_L+E(h_L)\right)^2+\sigma_{h_L}^2\right)} > \frac{\beta_H^2+2\beta_L\beta_H}{\beta_L^2}$$

The correspondence between the two sides of the inequality can be highlighted by separating the terms.

$$\frac{\left(\left(\beta_H+E(h_H)\right)^2+\sigma_{h_H}^2\right)}{\left(\left(\beta_L+E(h_L)\right)^2+\sigma_{h_L}^2\right)} + \frac{2\rho_{PL,PH}\sqrt{\left(\left(\beta_H+E(h_H)\right)^2+\sigma_{h_H}^2\right)}}{\sqrt{\left(\left(\beta_L+E(h_L)\right)^2+\sigma_{h_L}^2\right)}} > \frac{\beta_H^2}{\beta_L^2} + \frac{2\beta_H}{\beta_L} \tag{23.8}$$

Notice the correspondence between the high ideal beta plus the expected high *h* quantity squared plus the variance of the high *h* on the left-hand side and the high ideal beta squared on the right.

And the correspondence between the low ideal beta plus the expected low *h* quantity squared plus the variance of the low *h* on the left-hand side and the low ideal beta squared on the right.

This procedure can be continued for each condition, which removes from the conditions the market perturbation risk and leaves the questions of perturbation proportionality and correlation between perturbations.

The assessment of the nine full conditions and their relationship to the Sharpe ratios continues and is concluded in the next video.

Alternate derivation of (23.1)

If:

$$R_{PLt} = (\beta_L + h_{Lt})R_{PMt} \tag{22.3a}$$

Taking the variance of both sides:

$$\begin{aligned} var(R_{PL}) &= var\big((\beta_L + h_L)R_{PM}\big) = var(\beta_L R_{PM} + h_L R_{PM}) \\ &= var(\beta_L R_{PM}) + var(h_L R_{PM}) + 2\beta_L cov(R_{PM}, h_L R_{PM}) \end{aligned} \tag{23.9}$$

Each of the three terms on the right-hand side of equation (23.10) needs to evaluated separately.

The left-most term of the right-hand sided (23.10) is

$$var(\beta_L R_{PM}) = \beta_L^2 \sigma_{PM}^2 \tag{23.10}$$

The middle term on the right-hand side is the variance of a product of two random variables and can be expanded:

$$var(h_L R_{PM}) = \left(\sigma_{h_L}^2 + E(h_L)\right)(\sigma_{PM}^2 + E(h_L)^2) - \left(\sigma_{h_L,PM} + E(h_L)E(R_{PM})\right)^2 + cov(h^2, R_{PM}^2)$$

Since h_L and R_{PM} are independent, and $E(R_{PM}) = 0$, then $var(h_L R_{PM})$ reduces to

$$var(h_L R_{PM}) = \left(\sigma_{h_L}^2 + E(h_L)^2\right)\sigma_{PM}^2 \tag{23.11}$$

The right-hand term of (23.9) can be expanded:

$$\begin{aligned} 2\beta_L cov(R_{PM}, h_L R_{PM}) = 2\beta_L \Big(&\big(E(h_L)\sigma_{PM,PM} + E(R_{PM})\sigma_{h,PM}\big) + \\ &E\big[\big(R_{PM} - E(R_{PM})\big)\big(R_{PM} - E(R_{PM})\big)\big(h_L - E(h_L)\big)\big]\Big) \end{aligned}$$

Since h_L and R_{PM} are independent, this reduces to

$$2\beta_L cov(R_{PM}, h_L R_{PM}) = 2\beta_L E(h_L)\sigma_{PM}^2 \tag{23.12}$$

Adding the three pieces together [(23.11) + (23.12) + (23.13)] gives

$$\sigma_{PL}^2 = \beta_L^2 \sigma_{PM}^2 + \left(\sigma_{h_L}^2 + E(h_L)^2\right)\sigma_{PM}^2 + 2\beta_L E(h_L)\sigma_{PM}^2$$

The variance of the perturbations can be factored out.

$$\sigma_{PL}^2 = \left(\beta_L^2 + E(h_L)^2 + 2\beta_L E(h_L)^2 + \sigma_{h_L}^2\right)\sigma_{PM}^2$$

The first three terms in the parenthesis form a perfect square, and can be simplified.

$$\left(\beta_L^2 + E(h_L)^2 + 2\beta_L E(h_L) + \sigma_{h_L}^2\right) = \left(\beta_L + E(h_L)\right)^2 + \sigma_{h_L}^2 \tag{23.13}$$

(23.10) then reduces to

$$\sigma_{PL}^2 = \left(\left(\beta_L + E(h_L)\right)^2 + \sigma_{h_L}^2\right)\sigma_{PM}^2 \tag{23.2a}$$

$$\sigma_{PH}^2 = \left(\left(\beta_H + E(h_H)\right)^2 + \sigma_{h_H}^2\right)\sigma_{PM}^2 \tag{23.2b}$$

Where, necessarily

$$\sigma_{h_L}^2 = \sigma_{h_H}^2 \tag{23.3}$$

and

$$E(h_L) = -E(h_H) \tag{22.5}$$

Taking the square root of both sides of equation (23.2a) and (23.2b):

$$\sigma_{PL} = \sigma_{PM}\sqrt{\left(\beta_L + E(h_L)\right)^2 + \sigma_{h_L}^2} \tag{23.1a}$$

$$\sigma_{PH} = \sigma_{PM}\sqrt{\left(\beta_H + E(h_H)\right)^2 + \sigma_{h_H}^2} \tag{23.1b}$$

Alpha Found Video 24

Details and Generalizations Part 3

Welcome back again to part 24 of Alpha Found.

Let's assume for the moment that all the expected Sharpe ratios are equal.

$$E\left(\hat{S}'_M\right) = E\left(\hat{S}'_L\right) = E\left(\hat{S}'_H\right) \tag{9.5}$$

The equality of the two right-hand terms is scenario 6..

If we add noise in the form of $\sigma^2_{h_L} > 0$, then after σ^2_{PM} drops out, full condition 6 is:

$$\frac{\beta_H^2}{\beta_L^2} = \frac{\sigma^2_{PH}}{\sigma^2_{PL}} = \frac{\left(\beta_H + E(h_H)\right)^2 + \sigma^2_{h_H}}{\left(\beta_L + E(h_L)\right)^2 + \sigma^2_{h_L}} \tag{23.5}$$

Remember from the last video this condition can only exist if the expected high *h* is greater than zero, which is greater than the expected low *h*:

$$E(h_H) > 0 > E(h_L) \tag{23.6}$$

The equality of the two left-hand terms of (9.5) is scenario 8, and needs full condition 8 to have this as an expectation:

$$E\left(\hat{S}'_M\right) = E\left(\hat{S}'_L\right) = E\left(\hat{S}'_H\right) \tag{9.5}$$

Here is full condition 8:

$$\frac{\left(\beta_H + E(h_H)\right)^2 + \sigma^2_{h_H}}{\left(\beta_L + E(h_L)\right)^2 + \sigma^2_{h_L}} + \frac{2\rho_{PL,PH}\sqrt{\left(\beta_H + E(h_H)\right)^2 + \sigma^2_{h_H}}}{\sqrt{\left(\beta_L + E(h_L)\right)^2 + \sigma^2_{h_L}}} = \frac{\beta_H^2}{\beta_L^2} + \frac{2\beta_H}{\beta_L} \tag{24.1}$$

Note that there is a correlation term on the left-hand side.

We can solve for correlation from the decomposition of market perturbation risk.

$$\sigma^2_{PM} = \frac{\sigma^2_{PL}}{4} + \frac{\sigma^2_{PH}}{4} + \frac{2\rho_{PL,PH}\sigma_{PL}\sigma_{PH}}{2}$$

Multiplying through by four:

$$4\sigma^2_{PM} = \sigma^2_{PH} + \sigma^2_{PL} + 2\sigma_{PL,PH} = \sigma^2_{PH} + \sigma^2_{PL} + 2\rho_{PL,PH}\sigma_{PL}\sigma_{PH} \tag{24.2}$$

Substituting equations (23.2a) and (23.2b) gives:

$$4\sigma^2_{PM} = \left(\left(\beta_H + E(h_H)\right)^2 + \sigma^2_{h_H}\right)\sigma^2_{PM} + \left(\left(\beta_L + E(h_L)\right)^2 + \sigma^2_{h_L}\right)\sigma^2_{PM}$$
$$+2\rho_{PL,PH}\sqrt{\left(\left(\beta_H + E(h_H)\right)^2 + \sigma^2_{h_H}\right)\left(\left(\beta_L + E(h_L)\right)^2 + \sigma^2_{h_L}\right)}\,\sigma^2_{PM}$$

Market perturbation variance cancels from both sides, leaving an identity:

$$4 = \left(\left(\beta_H + E(h_H)\right)^2 + \sigma_{h_H}^2\right) + \left(\left(\beta_L + E(h_L)\right)^2 + \sigma_{h_L}^2\right) + 2\rho_{PL,PH}\sqrt{\left(\left(\beta_H + E(h_H)\right)^2 + \sigma_{h_H}^2\right)\left(\left(\beta_L + E(h_L)\right)^2 + \sigma_{h_L}^2\right)} \tag{24.3}$$

With simple algebra correlation is:

$$\rho_{PL,PH} = \frac{4 - \left(\left(\left(\beta_H + E(h_H)\right)^2 + \sigma_{h_H}^2\right) + \left(\left(\beta_L + E(h_L)\right)^2 + \sigma_{h_L}^2\right)\right)}{2\sqrt{\left(\left(\beta_H + E(h_H)\right)^2 + \sigma_{h_H}^2\right)\left(\left(\beta_L + E(h_L)\right)^2 + \sigma_{h_L}^2\right)}}$$

Which, with more algebra, can be simplified to:

$$\rho_{PL,PH} = \frac{\left(\beta_H + E(h_H)\right)\left(\beta_L + E(h_L)\right) - \sigma_{h_L}^2}{\sqrt{\left(\left(\beta_H + E(h_H)\right)^2 + \sigma_{h_H}^2\right)\left(\left(\beta_L + E(h_L)\right)^2 + \sigma_{h_L}^2\right)}} \tag{24.4}$$

There is another way to present the 9 full conditions. For example, from full condition 9, the expected Sharpe ratio of the low CAPM beta portfolio is greater than the expected market Sharpe ratio.

$$E\left(\hat{S}_L'\right) > E\left(\hat{S}_M'\right) \quad \rightarrow \frac{\beta_L\left(\bar{R}_M' - r_f\right)}{\sigma_L'} > \frac{\left(\bar{R}_M' - r_f\right)}{\sigma_M'} \quad \rightarrow \quad \frac{\beta_L}{\sigma_L'} > \frac{1}{\sigma_M'}$$

Squaring both sides, expanding, taking the reciprocal, and subtracting like terms gives:

$$\frac{\beta_L^2}{\beta_L^2\sigma_M^2 + \sigma_{PL}^2} > \frac{1}{\sigma_M^2 + \sigma_{PM}^2}$$

$$\sigma_M^2 + \frac{\sigma_{PL}^2}{\beta_L^2} < \sigma_M^2 + \sigma_{PM}^2 \quad \rightarrow \quad \frac{\sigma_{PL}^2}{\beta_L^2} < \sigma_{PM}^2$$

Moving the β_L^2 to the other side, making a substitution, and canceling σ_{PM}^2 from both sides gives:

$$\sigma_{PL}^2 < \beta_L^2\sigma_{PM}^2 \quad \rightarrow \quad \sigma_{PM}^2\left(\left(\beta_L + E(h_L)\right)^2 + \sigma_{h_L}^2\right) < \beta_L^2\sigma_{PM}^2$$

$$\left(\beta_L + E(h_L)\right)^2 + \sigma_{h_L}^2 < \beta_L^2 \tag{24.5}$$

Notice that since the variance of the low *h* is greater than zero ($\sigma_{h_L}^2 > 0$), the only way for inequality (24.5) to be true is if the expected low *h* is less than zero ($E(h_L) < 0$). If (24.5) were concerned with the high beta portfolio, then the inequality could not hold true as an expectation, but it might be true as a realization.

Expanding the left-hand side of (24.5) and subtracting the low ideal beta squared from both sides (β_L^2) gives:

$$\beta_L^2 + 2\beta_L E(h_L) + E(h_L)^2 + \sigma_{h_L}^2 < \beta_L^2 \quad \rightarrow \quad 2\beta_L E(h_L) + E(h_L)^2 + \sigma_{h_L}^2 < 0$$

Which becomes:

$$\sigma_{h_L}^2 < -2\beta_L E(h_L) - E(h_L)^2 \tag{24.6}$$

As a reminder, we are working toward understanding the conditions under which the expected Sharpe ratios have this order:

$$E(\hat{S}'_L) > E(\hat{S}'_M) > E(\hat{S}'_H).$$

Inequality (23.8) is, with algebra, the same as (24.5). That is (23.8):

$$\frac{\left(\left(\beta_H+E(h_H)\right)^2+\sigma_{h_H}^2\right)}{\left(\left(\beta_L+E(h_L)\right)^2+\sigma_{h_L}^2\right)} + \frac{2\rho_{PL,PH}\sqrt{\left(\left(\beta_H+E(h_H)\right)^2+\sigma_{h_H}^2\right)}}{\sqrt{\left(\left(\beta_L+E(h_L)\right)^2+\sigma_{h_L}^2\right)}} > \frac{\beta_H^2}{\beta_L^2} + \frac{2\beta_H}{\beta_L} \tag{23.8}$$

is algebraically equivalent to (24.5):

$$\left(\beta_L + E(h_L)\right)^2 + \sigma_{h_L}^2 < \beta_L^2 \tag{24.5}$$

It is not obvious that this is true, but you can verify it for yourself with some work. The first step is to substitute correlation from (24.4) into (23.8).

Look again at the equality of all Sharpe ratios:

$$E(\hat{S}'_M) = E(\hat{S}'_L) = E(\hat{S}'_H) \tag{9.5}$$

Only if the $\sigma_h^2 = 0$ and the $E(h_L) = 0$ can this equality hold. (See the proof of note six in video 22). Otherwise, the condition of the two right-hand terms is at odds with the condition of the two left-hand terms.

If the $\sigma_{h_L}^2 > 0$, then by definition h_L varies from one time period to the next, which also means the $\rho_{PL,PH} < 1$.

Here is the conclusion regarding Sharpe ratios:

$$\text{If } \sigma_{h_L}^2 > -2\beta_L E(h_L) - E(h_L)^2 \text{ then } E(\hat{S}'_L) < E(\hat{S}'_M) \tag{24.7}$$

$$\text{If } \sigma_{h_L}^2 = -2\beta_L E(h_L) - E(h_L)^2 \text{ then } E(\hat{S}'_L) = E(\hat{S}'_M) \tag{24.8}$$

$$\text{If } \sigma_{h_L}^2 < -2\beta_L E(h_L) - E(h_L)^2 \text{ then } E(\hat{S}'_L) > E(\hat{S}'_M) \tag{24.9}$$

(24.7) is consistent with conditions 1-7; (24.8) is consistent with condition 8, and (24.9) is consistent with condition 9.

Three quick numerical examples show how it works:

Let $\quad E(h_L) = -0.3$, and $\beta_L = 0.90$

then $\quad -2\beta_L E(h_L) - E(h_L)^2 = 0.45$

Since the correlation between perturbations has this formula:

$$\rho_{PL,PH} = \frac{(\beta_H+E(h_H))(\beta_L+E(h_L))-\sigma_{h_L}^2}{\sqrt{\left((\beta_H+E(h_H))^2+\sigma_{h_H}^2\right)\left((\beta_L+E(h_L))^2+\sigma_{h_L}^2\right)}} \tag{24.4}$$

Then this set of three numerical calculations follows:

If $\sigma_{h_L}^2 > 0.45$ then $\rho_{PL,PH} < 0.279135$ and $E(\hat{S}_L') < E(\hat{S}_M')$

If $\sigma_{h_L}^2 = 0.45$ then $\rho_{PL,PH} = 0.279135$ and $E(\hat{S}_L') = E(\hat{S}_M')$

If $\sigma_{h_L}^2 < 0.45$ then $\rho_{PL,PH} > 0.279135$ and $E(\hat{S}_L') > E(\hat{S}_M')$

If $\sigma_{h_L}^2 < 0.45$, then $\sigma_{h_L} < 0.67$. Subjectively, this seems to be an easy standard to meet for a diversified portfolio.

If we drop the magnitude of $E(h_L) = -0.3$ to $E(h_L) = -0.1$, then the needed σ_{h_L} drops from $\sigma_{h_L} < 0.671$ to $\sigma_{h_L} < 0.412$. Subjectively, this is also an easy standard to meet for a diversified portfolio, even with a small $E(h_L)$.

In other words, the market's attempt to have equal Sharpe ratios between the high and low CAPM beta portfolios, (full condition 6) takes only a small realized average high h greater than zero ($\bar{h}_H > 0$) to end up with full condition 9 instead. Condition 9 is a practical expectation. It takes very little to go from condition 6 to condition 9.

Therefore, under realistic, ordinary, and reasonable circumstances, and without altering arithmetic mean returns:

$$E(\hat{S}_L') > E(\hat{S}_M') > E(\hat{S}_H') \tag{13.2}$$

This concludes the discussion of the broad pattern of Sharpe ratios.

Now we have to return to alpha.

Alpha Found Video 25

Details and Generalizations Part 4

Welcome once again.

Let's review:

The low perturbation return in period *t* equals the low ideal beta plus the value of *h* in period *t* times the market perturbation return in period t.

$$R_{PLt} = (\beta_L + h_{Lt})R_{PMt} \tag{22.3a}$$

Although the sum of the *h*s is zero in any given time period:

$$\sum w_i(h_i) = 0 \tag{22.9}$$

we also saw that the individual expected *h*s do not equal 0 ($E(h_i) \neq 0$). If the expected *h* does equal 0 ($E(h_i) = 0$), then the expected return of the low CAPM beta portfolio in period *t*, given the perturbation return in period *t*, minus the risk-free rate, initially had this brief derivation:

$$\begin{aligned}
E(R'_{Lt}|R_{PMt}) - r_f &= \beta_L(R'_{Mt} - r_f) \\
&= \beta_L(R_{Mt} + R_{PMt} - r_f) \\
&= \beta_L(R_{Mt} - r_f) + \beta_L(R_{PMt}) \\
E(R'_{Lt}|R_{PMt}) - r_f &= \beta_L(R_{Mt} - r_f) + E(R_{PLt}|R_{PMt}) \qquad (22.2)
\end{aligned}$$

Where:

$$E(R_{PLt}|R_{PMt}) = \beta_L(R_{PMt}) \tag{22.1}$$

As discussed before, when $\sigma_h^2 > 0$ is included in the formula for perturbation risk, economic forces motivate the inclusion of a non-zero $E(h_i)$ for each portfolio, but the sum of the $E(h_i)$ for all portfolios is zero.

A complete return expectation conditioned on the perturbations, whether zero or non-zero is:

$$\begin{aligned}
E(R'_{Lt}|R_{PMt}) - r_f &= \beta_L(R'_{Mt} - r_f) + E(\hat{\alpha}_{PLt}|R_{PMt}) \\
&= \beta_L(R_{Mt} + R_{PMt} - r_f) + E(h_L)R_{PMt} \\
&= \beta_L(R_{Mt} - r_f) + \beta_L(R_{PMt}) + E(h_L)R_{PMt} \\
&= \beta_L(R_{Mt} - r_f) + (\beta_L + E(h_L))(R_{PMt}) \\
&= \beta_L(R_{Mt} - r_f) + E(R_{PLt}|R_{PMt}) \qquad (25.1)
\end{aligned}$$

Where:

$$E(R_{PLt}|R_{PMt}) = (\beta_L + E(h_{Lt}))(R_{PMt}) \tag{25.2}$$

Note the difference between (22.2) and (25.1):

$$E(R'_{Lt}|R_{PMt}) - r_f = \beta_L(R_{Mt} - r_f) + E(R_{PLt}) \tag{22.2}$$

Where:

$$E(R_{PLt}|R_{PMt}) = \beta_L(R_{PMt}) \tag{22.1}$$

$$E(R'_{Lt}|R_{PMt}) - r_f = \beta_L(R_{Mt} - r_f) + E(R_{PLt}) \tag{25.1}$$

Where:

$$E(R_{PLt}|R_{PMt}) = (\beta_L + E(h_L))(R_{PMt}) \tag{25.2}$$

The only difference is in the conditional expectations, where (22.1) does not have an *h* term while (25.2) does. (25.2) is the correct conditional expectation, (22.1) is not.

An alternate equivalent expression is, the expected return of the low CAPM beta portfolio in period *t* conditioned on the perturbation return minus the risk-free rate equals the ideal beta times the excess return of the market plus the expected perturbation alpha, also conditioned on the market perturbation return.

$$E(R'_{Lt}|R_{PMt}) - r_f = \beta_L(R'_{Mt} - r_f) + E(\hat{\alpha}_{PLt}|R_{PMt}) \tag{25.3}$$

Where the expected perturbation alpha in period *t*, conditioned on the perturbation return of the market in period *t* equals the expected *h* times the market perturbation return.

$$E(\hat{\alpha}_{PLt}|R_{PMt}) = E(h_{Lt})R_{PMt} \tag{25.4}$$

Which expression someone chooses, (25.1) or (25.3) is a matter of taste. My opinion is that (25.3) is more natural since it contains $R'_{Mt} - r_f$ rather than $R_{Mt} - r_f$.

Is positive α_P a good thing? This was covered in video 22, but I thought I would reiterate the point.

A negative *h* means a smaller positive perturbation when market perturbations are positive and a smaller negative perturbation when market perturbations are negative. For a long-term investor this means lower volatility for the same arithmetic mean return, which in turn means a higher CAPM alpha, a higher Sharpe ratio, and a higher geometric return than other investors with the same arithmetic mean return but with a positive *h*.

In video 12, the formula for conditional expected CAPM alpha was:

$$E(\hat{\alpha}'_L|R_{PM} = R_{PL} = 0) = (\beta_L - \beta'_L)(\bar{R}'_M - r_f) \tag{12.6a}$$

More generally, expected CAPM alpha should be written like this:

$$E(\hat{\alpha}'_L|\bar{R}_{PM}) = (\beta_L - \beta'_L)(\bar{R}'_M - r_f) + E(h_L)\bar{R}_{PM} \tag{25.5}$$

Realized CAPM alpha has this form:

$$\hat{\alpha}'_L = (\hat{\beta}_L - \hat{\beta}'_L)(\bar{R}'_M - r_f) + \frac{1}{T}\sum h_{Lt} R_{PMt} \tag{25.6}$$

Or alternatively:

$$\hat{\alpha}_L' = \left(\hat{\beta}_L - \hat{\beta}_L'\right)\left(\bar{R}_M' - r_f\right) + \bar{\alpha}_{PL} \tag{25.7}$$

This more general formula includes the possibility that $\bar{R}_{PM} = 0$, while $\bar{\alpha}_{PL} = -\bar{\alpha}_{PH} \neq 0$.

This concludes the discussion of the broad patterns of CAPM alpha.

Here is the overall conclusion:

The consistency of these two realizations in a generally rising market:

$$\hat{\alpha}_L' > 0 > \hat{\alpha}_H' \tag{9.6}$$

$$\hat{S}_L' > \hat{S}_M' > \hat{S}_H' \tag{9.7}$$

Has been shown to be the result of these two expectations:

$$E(\hat{\alpha}_L') > 0 > E(\hat{\alpha}_H') \tag{13.1}$$

$$E(\hat{S}_L') > E(\hat{S}_M') > E(\hat{S}_H') \tag{13.2}$$

Which are natural to the structure of stock market return data without reference to financial analysis, accounting data, or economic forecasts.

It was the stated in video 9, "to the extent that (9.6) and (9.7) can be expressed as mathematical expectations ex-ante, the volatility effect is resolved." The goal of this series of videos has been reached.

We still need to talk about individual stocks.

Alpha Found Video 26

The Forward Perturbation Risk Model for Individual Stocks

Welcome once again to Alpha Found.

The forward perturbation risk model for an individual stock follows the same pattern as for the high and low CAPM beta portfolios. Therefore, there is less deriving than in some earlier videos and more simply acknowledging.

Here are the familiar formulas, in the context of an individual stock.

$$\text{PRM: } E(\bar{R}'_i) - r_f = \beta_i(\bar{R}'_M - r_f) \tag{26.1}$$

$$\text{CAPM: } E(\bar{R}'_i) - r_f = E(\hat{\alpha}'_i) + \beta'_i(\bar{R}'_M - r_f) \tag{26.2}$$

$$E(\hat{\alpha}'_i | \bar{R}_{PM} = \bar{R}_{Pi} = 0) = (\beta_i - \beta'_i)(\bar{R}'_M - r_f) \tag{26.3}$$

$$E(\hat{\alpha}'_i | \bar{R}_{PM}) = (\beta_i - \beta'_i)(\bar{R}'_M - r_f) + E(h_i)\bar{R}_{PM} \tag{26.4}$$

$$\hat{\alpha}'_i = (\hat{\beta}_i - \hat{\beta}'_i)(\bar{R}'_M - r_f) + \frac{1}{T}\sum h_{it}R_{Pit} = (\hat{\beta}_i - \hat{\beta}'_i)(\bar{R}'_M - r_f) + \bar{\alpha}_{Pi} \tag{26.5}$$

$$E(\bar{R}_{Pi} | \bar{R}_{PM}) = (\beta_i + E(h_i))\bar{R}_{PM} \tag{26.6}$$

$$\sigma_{Pi} = \sigma_{PM}\sqrt{(\beta_i + E(h_i))^2 + \sigma^2_{h_i}} \tag{26.7}$$

The formula for the expected CAPM alpha of an individual stock is in equation (26.4).

$$E(\hat{\alpha}'_i | \bar{R}_{PM}) = (\beta_i - \beta'_i)(\bar{R}'_M - r_f) + E(h_i)\bar{R}_{PM} \tag{26.4}$$

Next, we derive β'_i in the context of PRM.

$$cov(R'_i, R'_M) = cov(R_i + R_{Pi}, R_M + R_{PM})$$

$$= cov(R_i, R_M) + cov(R_i, R_{PM}) + cov(R_{Pi}, R_M) + cov(R_{Pi}, R_{PM})$$

$$= cov(R_i, R_M) + cov(R_{Pi}, R_{PM})$$

With a change of notation:

$$= \beta_i\sigma^2_M + \sigma_{Pi,PM}$$

The covariance between an individual stock and the market is:

$$\sigma'_{i,M} = \beta_i\sigma^2_M + \sigma_{Pi,PM} \tag{26.8}$$

Since CAPM beta is covariance divided by market variance, then CAPM beta is:

$$\beta_i' = \frac{\sigma_{i,M}'}{\sigma_M'^2} = \frac{\beta_i\sigma_M^2 + \sigma_{Pi,PM}}{\sigma_M^2 + \sigma_{PM}^2} \tag{26.9}$$

$$= \frac{\beta_i\sigma_M^2 + \rho_{Pi,PM}\sigma_{Pi}\sigma_{PM}}{\sigma_M^2 + \sigma_{PM}^2}$$

$$= \frac{\beta_i\sigma_M^2 + \rho_{Pi,PM}\sqrt{\left(\beta_i + E(h_i)\right)^2 + \sigma_{h_i}^2}\,\sigma_{PM}^2}{\sigma_M^2 + \sigma_{PM}^2}$$

Which gives us an alternate formulation for CAPM beta.

$$\beta_i' = \frac{\beta_i\sigma_M^2 + \rho_{Pi,PM}\sqrt{\left(\beta_i + E(h_i)\right)^2 + \sigma_{h_i}^2}\,\sigma_{PM}^2}{\sigma_M^2 + \sigma_{PM}^2} \tag{26.10}$$

Note that if $\beta_i = \rho_{Pi,PM}\sqrt{\left(\beta_i + E(h_i)\right)^2 + \sigma_{h_i}^2}$, and this is substituted into equation (26.10), we get this nice result:

$$\beta_i' = \frac{\beta_i\sigma_M^2 + \beta_i\sigma_{PM}^2}{\sigma_M^2 + \sigma_{PM}^2} = \frac{\beta_i(\sigma_M^2 + \sigma_{PM}^2)}{\sigma_M^2 + \sigma_{PM}^2} = \beta_i$$

CAPM beta equals ideal beta. This shows once again how unlikely it is for CAPM conditions to be met.

To have an expected positive CAPM alpha in a generally rising market with zero average perturbations requires:

$$E(\hat{\alpha}_i' | \bar{R}_{PM} = \bar{R}_{Pi} = 0) = (\beta_i - \beta_i')\left(\bar{R}_M' - \overline{r_f}\right) > 0 \tag{26.3}$$

Which means ideal beta is greater than CAPM beta.

$$\beta_i > \beta_i' = \frac{\beta_i\sigma_M^2 + \sigma_{Pi,PM}}{\sigma_M^2 + \sigma_{PM}^2}$$

Which, with algebra becomes:

$$\beta_i > \rho_{Pi,PM}\sqrt{\left(\beta_i + E(h_i)\right)^2 + \sigma_{h_i}^2}$$

This gives us 3 possibilities all derived similarly.

If ideal beta is greater than CAPM beta, then expected alpha is positive:

$$\beta_i > \rho_{Pi,PM}\sqrt{\left(\beta_i + E(h_i)\right)^2 + \sigma_{h_i}^2} \quad \text{then} \quad E(\hat{\alpha}_i' | \bar{R}_{PM} = \bar{R}_{Pi} = 0) > 0 \tag{26.11}$$

If ideal beta equals CAPM beta, then expected alpha is zero:

$$\beta_i = \rho_{Pi,PM}\sqrt{\left(\beta_i + E(h_i)\right)^2 + \sigma_{h_i}^2} \quad \text{then} \quad E(\hat{\alpha}_i' | \bar{R}_{PM} = \bar{R}_{Pi} = 0) = 0 \tag{26.12}$$

or if ideal beta is less than CAPM beta, expected alpha is negative:

$$\beta_i < \rho_{Pi,PM}\sqrt{\left(\beta_i + E(h_i)\right)^2 + \sigma_{h_i}^2} \;\text{ then }\; E(\hat{\alpha}_i' | \bar{R}_{PM} = \bar{R}_{Pi} = 0) < 0 \tag{26.13}$$

On the subject of individual Sharpe ratios, here is a quick derivation, assuming $\bar{R}_{PM} = \bar{R}_{Pi} = 0$:

$$E(\hat{S}_i') > E(\hat{S}_M') \quad \rightarrow \quad \frac{\beta_i(R_M' - r_f)}{\sqrt{\beta_i^2\sigma_M^2 + \sigma_{Pi}^2}} > \frac{(R_M' - r_f)}{\sqrt{\sigma_M^2 + \sigma_{PM}^2}}$$

Divide through by excess return, and square both sides:

$$\frac{\beta_i}{\sqrt{\beta_i^2\sigma_M^2 + \sigma_{Pi}^2}} > \frac{1}{\sqrt{\sigma_M^2 + \sigma_{PM}^2}} \quad \rightarrow \quad \frac{\beta_i^2}{\beta_i^2\sigma_M^2 + \sigma_{Pi}^2} > \frac{1}{\sigma_M^2 + \sigma_{PM}^2}$$

Take the reciprocal of both sides, cancel, and subtract like terms:

$$\frac{\beta_i^2\sigma_M^2 + \sigma_{Pi}^2}{\beta_i^2} < \sigma_M^2 + \sigma_{PM}^2 \quad \rightarrow \quad \sigma_M^2 + \frac{\sigma_{Pi}^2}{\beta_i^2} < \sigma_M^2 + \sigma_{PM}^2 \quad \rightarrow \quad \frac{\sigma_{Pi}^2}{\beta_i^2} < \sigma_{PM}^2$$

$$\sigma_{Pi}^2 < \beta_i^2\sigma_{PM}^2 \quad \rightarrow \quad \sigma_{PM}^2\left(\left(\beta_i + E(h_i)\right)^2 + \sigma_{h_i}^2\right) < \beta_i^2\sigma_{PM}^2$$

$$\left(\beta_i + E(h_i)\right)^2 + \sigma_{h_i}^2 < \beta_i^2 \tag{26.14}$$

There is a similar derivation for $E(\hat{S}_i') = E(\hat{S}_M')$ and $E(\hat{S}_i') < E(\hat{S}_M')$.

The result can be seen here:

$$\text{If } \left(\beta_i + E(h_i)\right)^2 + \sigma_{h_i}^2 < \beta_i^2 \text{ then } E(\hat{S}_i') > E(\hat{S}_M') \tag{26.15}$$

$$\text{If } \left(\beta_i + E(h_i)\right)^2 + \sigma_{h_i}^2 = \beta_i^2 \text{ then } E(\hat{S}_i') = E(\hat{S}_M') \tag{26.18}$$

$$\text{If } \left(\beta_i + E(h_i)\right)^2 + \sigma_{h_i}^2 > \beta_i^2 \text{ then } E(\hat{S}_i') < E(\hat{S}_M') \tag{26.19}$$

An individual stock will likely have a higher value for $\sigma_{h_i}^2$ than a diversified portfolio, therefore an individual stock will likely have a lower Sharpe ratio than the market portfolio.

For a single stock there is a 5-scenario analog to the 9 scenarios we have previously considered.

Table 26.1

Expected Relationship Between OLS Alpha and Sharpe Ratios

For Different $E(h_i)$ and $\sigma_{h_i}^2$ Relationships

	Expected $\widehat{\alpha}'$	Expected $\widehat{S}'$	Condition	
1	$E(\hat{\alpha}_i') < 0$	$E(\hat{S}_i') < E(\hat{S}_M')$	$(\beta_i + E(h_i))^2 + \sigma_{h_i}^2 \geq \rho_{Pi,PM}^2 \left((\beta_i + E(h_i))^2 + \sigma_{h_i}^2\right) > \boldsymbol{\beta_i^2}$	(26.15)
2	$E(\hat{\alpha}_i') = 0$	$E(\hat{S}_i') < E(\hat{S}_M')$	$(\beta_i + E(h_i))^2 + \sigma_{h_i}^2 > \rho_{Pi,PM}^2 \left((\beta_i + E(h_i))^2 + \sigma_{h_i}^2\right) = \boldsymbol{\beta_i^2}$	(26.16)
3	$E(\hat{\alpha}_i') > 0$	$E(\hat{S}_i') < E(\hat{S}_M')$	$(\beta_i + E(h_i))^2 + \sigma_{h_i}^2 > \boldsymbol{\beta_i^2} > \rho_{Pi,PM}^2 \left((\beta_i + E(h_i))^2 + \sigma_{h_i}^2\right)$	(26.17)
4	$E(\hat{\alpha}_i') > 0$	$E(\hat{S}_i') = E(\hat{S}_M')$	$(\beta_i + E(h_i))^2 + \sigma_{h_i}^2 = \boldsymbol{\beta_i^2} > \rho_{Pi,PM}^2 \left((\beta_i + E(h_i))^2 + \sigma_{h_i}^2\right)$	(26.18)
5	$E(\hat{\alpha}_i') > 0$	$E(\hat{S}_i') > E(\hat{S}_M')$	$\boldsymbol{\beta_i^2} > (\beta_i + E(h_i))^2 + \sigma_{h_i}^2 \geq \rho_{Pi,PM}^2 \left((\beta_i + E(h_i))^2 + \sigma_{h_i}^2\right)$	(26.19)

The red color is simply to highlight the location of the ideal beta squared.

I have made an effort to develop a backward perturbation risk model for an individual stock. So far it has been intractable. It awaits further insights.

Alpha Found Video 27

Loose Ends from Video 10

Welcome once again to part 27.

In video 10, called Setting the Table Part 2, there were 13 two-year time periods where two securities were selected randomly and combined to form an index. Each of the two securities were then regressed against the index. The statistics of CAPM alpha, beta, and Sharpe ratio were also collected.

This was repeated 5,000 times for each two-year time period. Also, a period when the market was generally falling was used. That period was shorter than two years.

Here is the result as shown in video 10.

Table 10.1
Summary Statistics for Random Pairs of Stocks for 14 time periods

	1994-1995	1996-1997	1998-1999	2000-2001	2002-2003	2004-2005	2006-2007	2008-2009	2010-2011	2012-2013	2014-2015	2016-2017	2018-2019	Down
Number of pairs	4,996	4,998	5,000	5,000	4,998	4,998	5,000	4,998	4,999	4,997	4,997	4,999	4,999	4,991
Low Beta-High Alpha	3,332	3,782	2,200	3,148	3,486	3,200	2,846	2,505	3,157	3,744	3,196	3,628	3,261	1,998
% Low Beta-High Alpha	66.69%	75.67%	44.00%	62.96%	69.75%	64.03%	56.92%	50.12%	63.15%	74.92%	63.96%	72.57%	65.23%	40.03%
Average α' for Low β'	0.404%	0.900%	-0.232%	0.466%	0.658%	0.416%	0.208%	-0.026%	0.376%	0.798%	0.348%	0.652%	0.360%	-0.486%
Average Low Beta	0.6372	0.5727	0.6013	0.5505	0.5488	0.5735	0.5898	0.6529	0.6461	0.5627	0.5843	0.5643	0.6015	0.6872
Avg. Market Return	1.26%	2.14%	1.05%	1.40%	2.08%	1.38%	0.71%	0.36%	1.07%	2.40%	0.16%	1.63%	0.77%	-3.96%
Sharpe Ratio order														
Low-Market-High	2,277	2,503	1,636	2,201	2,539	2,282	2,176	2,056	2,514	2,347	2,601	2,497	2,671	1,284
Low-High-Market	37	114	213	101	16	89	180	155	57	26	206	35	91	1,104
Market-Low-High	613	817	229	517	518	459	306	196	361	836	268	671	270	-
Market-High-Low	944	823	431	724	817	827	564	363	552	1,084	439	901	512	7
High-Market-Low	1,103	729	2,351	1,401	1,089	1,308	1,690	2,127	1,474	699	1,385	885	1,406	1,842
High-Low-Market	22	12	140	56	19	33	84	101	41	5	98	10	49	754
Sharpe Ratio order														
% Low-Market-High	45.58%	50.08%	32.72%	44.02%	50.80%	45.66%	43.52%	41.14%	50.29%	46.97%	52.05%	49.95%	53.43%	25.73%
% Low-High-Market	0.74%	2.28%	4.26%	2.02%	0.32%	1.78%	3.60%	3.10%	1.14%	0.52%	4.12%	0.70%	1.82%	22.12%
% Market-Low-High	12.27%	16.35%	4.58%	10.34%	10.36%	9.18%	6.12%	3.92%	7.22%	16.73%	5.36%	13.42%	5.40%	0.00%
% Market-High-Low	18.90%	16.47%	8.62%	14.48%	16.35%	16.55%	11.28%	7.26%	11.04%	21.69%	8.79%	18.02%	10.24%	0.14%
% High-Market-Low	22.08%	14.59%	47.02%	28.02%	21.79%	26.17%	33.80%	42.56%	29.49%	13.99%	27.72%	17.70%	28.13%	36.91%
% High-Low-Market	0.44%	0.24%	2.80%	1.12%	0.38%	0.66%	1.68%	2.02%	0.82%	0.10%	1.96%	0.20%	0.98%	15.11%

For 11 of the 14 time periods the results were consistent with the predictions of the perturbation risk model, where $\bar{R}_{PM} \approx 0$. Regarding the three time periods that were not consistent, I said "For the time being I am going to pretend I didn't notice." They are highlighted with big red dots on the following graph.

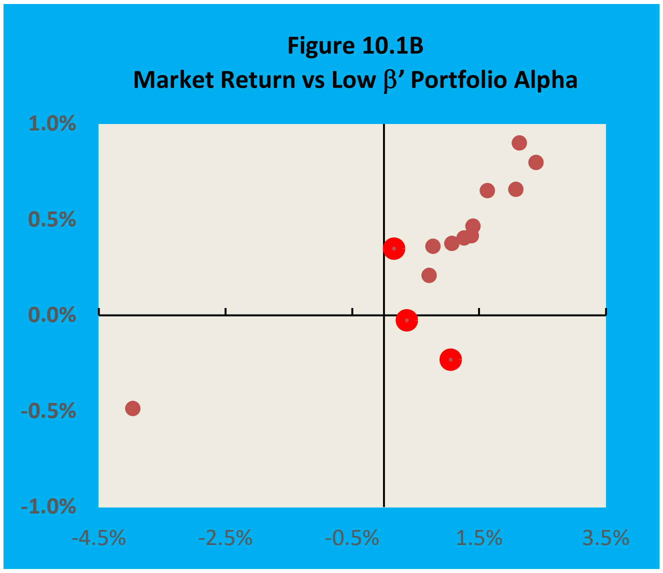

Table 10.2B

	Market Return	Average Alpha
1994-1995	1.264%	0.404%
1996-1997	2.136%	0.900%
1998-1999	1.052%	-0.232%
2000-2001	1.404%	0.466%
2002-2003	2.078%	0.658%
2004-2005	1.380%	0.416%
2006-2007	0.714%	0.208%
2008-2009	0.360%	-0.026%
2010-2011	1.072%	0.376%
2012-2013	2.396%	0.798%
2014-2015	0.158%	0.348%
2016-2017	1.632%	0.652%
2018-2019	0.774%	0.360%
Down Market	-3.964%	-0.486%

The intent of this video is to show loosely how the anomalous results in those three time periods might get resolved within the perturbation risk model. No high rigor solutions are offered since there is not enough information to construct such a rigorous solution.

The original method of data analysis was a little suspect, but since it is only suggestive, I suggested that it's good enough for our purposes.

One thesis from the earlier videos is that the $E(\hat{\alpha}_L')$, when average market perturbations are zero, equals the low ideal beta estimate, minus the low CAPM beta estimate, times average excess return.

$$E(\hat{\alpha}_L' | \bar{R}_{PM} = \bar{R}_{PL} = 0) = \left(\hat{\beta}_L - \hat{\beta}_L'\right)\left(\bar{R}_M' - r_f\right) \tag{12.6a}$$

If the market is rising then the average excess return is greater than zero.

$$\left(\bar{R}_M' - r_f\right) > 0$$

And therefore, the expected low CAPM alpha is greater than zero.

$$E(\hat{\alpha}'_L | \bar{R}_{PM} = \bar{R}_{Pi} = 0) > 0$$

It didn't work out that way for the results in 1998 – 1999, or 2008 – 2009, where the $\hat{\alpha}'_L < 0$.

A formula more realistic than (12.6a) is the ex-post version of (25.5) shown here:

$$E(\hat{\alpha}'_L | \bar{R}_{PM}) = (\hat{\beta}_L - \hat{\beta}'_L)(\bar{R}'_M - r_f) + \bar{h}_L \bar{R}_{PM} \tag{27.1}$$

The results from 1998-1999 have the following statistical components:

$$E(\hat{\alpha}'_L) = -0.232\% = (\hat{\beta}_L - 0.6013)(1.052\%) + \bar{h}_L \bar{R}_{PM}$$

In the usual case $(\hat{\beta}_L - \hat{\beta}'_L) > 0$.

If the $(\hat{\beta}_L - \hat{\beta}'_L) = 0.24$, then plugging in we get:

$$-0.232\% = (.24)(1.052\%) + \bar{h}_L \bar{R}_{PM}$$

Which leads to $\bar{h}_L$ times $\bar{R}_{PM}$ equal to -0.4845%:

$$-0.4845\% = \bar{h}_L \bar{R}_{PM}$$

Using the modelling from video 24 it is expected that:

$$\bar{h}_L < 0$$

Based on information not yet in a video, assume:

$$\bar{h}_L = -0.42$$

This number is somewhat arbitrary and has no specific importance other than it is below zero and in right range.

Which leads to the average market perturbation equal to 1.154%.

$$-0.4845\% = \bar{h}_L \bar{R}_{PM} \quad \rightarrow \quad \bar{R}_{PM} = \frac{-0.4845\%}{\bar{h}_L} = \frac{-0.4845\%}{-0.42} = 1.154\%$$

Which means the average low h is -0.42, the average market perturbation is 1.154%, and the average excess ideal market return is -0.102%.

$$\bar{h}_L = -0.42, \quad \bar{R}_{PM} = 1.154\%, \quad \bar{R}_M - r_f = -0.102\%$$

For 1998-1999 this is one possible way to make the statistics fit the model.

If the difference between the low ideal beta estimate and the low CAPM beta estimate is different from the guess made here, or $\bar{h}_L$ is different, or both, then $\bar{R}_{PM}$ is also different.

The statistical components of the results from 2008-2009 are analyzed similarly.

$$E(\hat{\alpha}_L') = -0.026\% = \left(\hat{\beta}_L - 0.6529\right)(0.360\%) + \bar{h}_L \bar{R}_{PM}$$

Setting $\left(\hat{\beta}_L - \hat{\beta}_L'\right) = 0.24$ again:

$$-0.026\% = (.24)(0.360\%) + \bar{h}_L \bar{R}_{PM}$$

which leads to average low h times the average market perturbation equal to -0.1124%.

$$-0.1124\% = \bar{h}_L \bar{R}_{PM}$$

Again using:

$$\bar{h}_L = -0.42$$

which leads to the average market perturbation equal to 0.2676%.

$$-0.1124\% = \bar{h}_L \bar{R}_{PM} \quad \rightarrow \quad \bar{R}_{PM} = \frac{-0.1124\%}{\bar{h}_L} = \frac{-0.1124}{-0.42} = 0.2676\%$$

Which means:

$$\bar{h}_L = -0.42, \quad \bar{R}_{PM} = 0.2676\%, \quad \bar{R}_M - r_f = 0.0924\%$$

For 2008-2009 this is one possible way to make the statistics fit the model.

If the difference between low ideal and CAPM betas is different from the guess made here, or $\bar{h}_L$ is different, or both, then $\bar{R}_{PM}$ is different as well.

Finally, for 2014-2015 the statistics are as shown:

$$E(\hat{\alpha}_L') = 0.348\% = \left(\hat{\beta}_L - 0.5843\right)(0.158\%) + \bar{h}_L \bar{R}_{PM}$$

Setting $\left(\hat{\beta}_L - \hat{\beta}_L'\right) = 0.24$, again:

$$0.348\% = (.24)(0.158\%) + \bar{h}_L \bar{R}_{PM}$$

Which leads to:

$$\bar{h}_L \bar{R}_{PM} = 0.3101\%$$

Again using:

$$\bar{h}_L = -0.42$$

Which leads to $\bar{R}_{PM} = -0.7383\%$.

$$0.3101\% = \bar{h}_L \bar{R}_{PM} \quad \rightarrow \quad \bar{R}_{PM} = \frac{0.3101\%}{\bar{h}_L} = \frac{0.3101\%}{-0.42} = -0.7383\%$$

Which means:

$$\bar{h}_L = -0.42, \quad \bar{R}_{PM} = -0.7383\%, \quad \bar{R}_M - r_f = 0.8963\%$$

For 2014-2015 this is one possible way to make the statistics fit the model.

If the difference between low ideal and CAPM betas is different from the guess made here, or $\bar{h}_L$ is different, or both, then $\bar{R}_{PM}$ is also different.

There is no suggestion that the results from these examples are correct. The only point is that when the results don't fit well with the assumption $\bar{R}_{PM} \approx 0$, then it is easy to find a different assumption that will fit.

This is both a strength and a weakness. It's a strength because of the inherent flexibility of the model, and a weakness because it makes the model harder to falsify, if it is indeed false.

Alpha Found Video 28

Summary and Additional Insights

Welcome.

The modeling explored in these videos places the volatility effect and other issues in a new light.

The persistent high $\hat{\alpha}'_L$s for low CAPM beta portfolios over time are likely a statistical artifact of ordinary least squares regression rather than a true economic anomaly, while the high Sharpe ratios of low CAPM beta portfolios are likely to be an ongoing phenomenon.

Like the capital asset pricing model, the perturbation risk model has two components: Risk and Return.

Return has two components: Ideal return and perturbation return. Risk has two components: ideal risk and perturbation risk.

Two Sources of Return	**Two Sources of Risk**
Ideal Return	Ideal Risk
Perturbation Return	Perturbation Risk

Ideal returns are pure economic fair value returns.

Perturbation returns are returns not accounted for by changes in fair value. They are expected to be zero and are mean reverting. They are analogous to or the same as excess volatility.

In the modeling of these videos, the volatility effect occurs as a result of the distribution of perturbation risk and how the distribution coincides or differs from the distribution of ideal risk. That is, the volatility effect is the result of the magnitude and distribution of the perturbation risk among securities.

The definition of the volatility effect for this series of videos was given as:

Given these two expectations:

$$E(\hat{\alpha}'_L) = E(\hat{\alpha}'_H) = 0 \tag{9.4}$$

$$E\left(\hat{S}'_L\right) = E\left(\hat{S}'_H\right) = E\left(\hat{S}'_M\right) \tag{9.5}$$

The volatility effect is the consistency of these two realizations in a generally rising market.

$$\hat{\alpha}'_L > 0 > \hat{\alpha}'_H \tag{9.6}$$

$$\hat{S}'_L > \boldsymbol{\hat{S}'_M} > \hat{S}'_H \tag{9.7}$$

In video 9 it was stated “To the extent that (9.6) and (9.7) can be expressed as mathematical expectations (ex-ante) the volatility effect is resolved.”

In the perturbation risk model, inequality (9.6) has been shown to be a mathematical expectation, and inequality (9.7) has been shown to be a practical expectation, or possibly a mathematical expectation depending on how conditions are interpreted. (9.6) and (9.7) are realized because full condition 9 in line

(21.9) is met, and so both parts of scenario 9 are expected.

$$E(\hat{\alpha}'_L) > 0 > E(\hat{\alpha}'_H) \tag{21.9}$$

$$E(\hat{S}'_L) > \boldsymbol{E(\hat{S}'_M)} > E(\hat{S}'_H) \tag{21.9}$$

Separating out ideal risk from perturbation risk gives many new formulas, techniques, and insights. It also gives many new problems to investigate.

Perturbation risk itself has two components:

> That which contributes to market risk or traditional systematic risk and,
>
> That which sums to zero across all securities in each time period, called perturbation alpha, although the component perturbation alphas do not have a zero expected value when conditioned on a non-zero perturbation.

The components of CAPM are α', β', R'_M, σ'^2_M, σ'^2_ε, and their combinations.

The components of PRM are α_P, β, R_M, σ^2_M, R_{PM}, σ^2_{PM}, and their combinations.

There is no simple correspondence between the components of CAPM and the components of PRM.

CAPM beta now has three equivalent formulas:

$$\beta'_i = \frac{\sigma'_{i,M}}{\sigma'^2_M} = \rho'_{i,M}\frac{\sigma'_i}{\sigma'_M} \tag{6.1}$$

$$\beta'_i = \frac{\beta_i\sigma^2_M + \sigma_{Pi,PM}}{\sigma'^2_M} \tag{26.9}$$

$$\beta_i = \frac{\beta_i\sigma^2_M + \rho_{Pi,PM}\sqrt{\left(\beta_i + E(h_i)\right)^2 + \sigma^2_{h_i}}\,\sigma^2_{PM}}{\sigma^2_M + \sigma^2_{PM}} \tag{26.10}$$

Also, with respect to the original problem formulation, CAPM beta for the low CAPM beta portfolio has this formula:

$$\beta'_L = \frac{\beta_L\sigma^2_M + \frac{1}{2}\sigma^2_{PL} + \frac{1}{2}\sigma_{PL,PH}}{\sigma'^2_M} \tag{12.21a}$$

Here are the five possible combinations of perturbation attributes.

Table 28.1
Five Possible Combinations of Perturbations Attributes

	Perturbation attributes			Equivalent Attributes of h_i	
1.	$R_{PMt} = 0$	$\sigma_{Pi} = 0,$	$\rho_{Pi,PM} =$ undefined	$h_{it} = 0,\quad \sigma^2_{h_i} = 0$	(28.1)
2.	$R_{PMt} \neq 0$	$\sigma_{Pi} = \beta_i\sigma_{PM}$	$\rho_{Pi,PM} = 1$	$h_{it} = 0,\quad \sigma^2_{h_i} = 0$	(28.2)
3.	$R_{PMt} \neq 0$	$\sigma_{Pi} \neq \beta_i\sigma_{PM}$	$\rho_{Pi,PM} = 1$	$E(h_i) \neq 0,\ \sigma^2_{h_i} = 0$	(28.3)

4. $R_{PMt} \neq 0 \quad \sigma_{Pi} = \beta_i \sigma_{PM} \quad \rho_{Pi,PM} \neq 1 \qquad E(h_i) = 0,\ \sigma_{h_i}^2 \neq 0 \qquad (28.4)$
5. $R_{PMt} \neq 0 \quad \sigma_{Pi} \neq \beta_i \sigma_{PM} \quad \rho_{Pi,PM} \neq 1 \qquad E(h_i) \neq 0,\ \sigma_{h_i}^2 \neq 0 \qquad (28.5)$

Combination 2 is note 6 in video 22.

Combination 3 has non-zero alpha but no idiosyncratic risk.

Combination 5 is the real world.

There are many places in the videos where lesser problems were left unresolved, details were skipped, or some problems were even completely unmentioned.

For instance, in video 9, the structure of correlation in beta deciles is almost surely correct, but a proof was not pursued. An analysis of this and related issues can be found in videos 30 and 31.

Sometimes the algebraic development of an equation was left out because it seemed to be a distraction from more important points.

One unmentioned aspect is the fact that although the low CAPM beta is lower than the high CAPM beta, it is not necessarily the case that the low ideal beta, associated with the low CAPM beta portfolio, is lower than the high ideal beta, associated with the high CAPM beta portfolio. The order can be reversed.

Here are some additional areas in need of further development:

1. At a practical level, what are the implications of the perturbation risk model for short selling, diversification, and short term versus long term investment strategies?

   Different investment strategies result from having different investment time frames. Different time frames may well put investors at odds with each other, with some investors seeking negative h and others seeking positive h. A long-term investor would seek to minimize h in order to maximize geometric return. A single time frame investor might seek a high h if high market returns are anticipated, or seek a low h if negative market returns are anticipated, or perhaps use short selling or exit the market if investment policy allows such a response when low returns are anticipated.

   The existence of "high h" investors could create a self-fulfilling result as each investor tries to beat the other investors in to or out of the door. This could influence institutional pressure on managers of mutual funds who seek impressive short-term results.

   So, this is one area in need of further development. Here are some additional areas also in need of development.

2. The backward PRM for an individual stock is lacking.

3. The estimation method for ideal beta needs to be improved.

4. The causes of the magnitude of the perturbations should be developed.

5. The methods to estimate the magnitude of excess volatility needs improvement.

6. What are the implications of the perturbation risk model for metrics such as tracking error?

7. It's clear in the perturbation risk model that the market portfolio is not on the efficient frontier. What does the Capital Market Line look like in the perturbation risk model, and what does it mean for efficient market theory?

8. Is the equality of Sharpe ratios the best meaning of market efficiency in the PRM?

9. The zero-beta portfolio should be reexamined because current methods likely have some residual ideal systematic risk.

10. What are the implications for the PRM if the variables of ideal return and perturbation return are non-normal, perhaps log-normal? Returns would then be the sum of two log-normal random variables, perhaps correlated log-normal random variables. The mathematics is no longer so straightforward, but perhaps more realistic.

11. What are the broader implications for OLS if the independent variable is the sum of two random variables that cannot be easily separated such as signal plus accumulated noise?

Many things have been left unexplored; I have not intended to answer every question, but only lay out the main arguments.

I think I'll stop here.

Alpha Found Video 29

Missing Algebra

Welcome back to an unplanned episode of Alpha Found.

Several times in video 24, I used the words "with algebra" or "with simple algebra," etc. without showing steps, since many of the steps are not obvious, I thought it would be best to fill in the details. Also, there are two additional identities, one of which was left out of video 24 because I felt it was time consuming and irrelevant. Now I think it's a mistake to leave it out. It's not irrelevant.

We'll begin with some simple algebra to solve for correlation. Recall identity (24.3):

$$4 = \left(\left(\beta_H + E(h_H)\right)^2 + \sigma_{h_H}^2\right) + \left(\left(\beta_L + E(h_L)\right)^2 + \sigma_{h_L}^2\right)$$

$$+2\rho_{PL,PH}\sqrt{\left(\left(\beta_H + E(h_H)\right)^2 + \sigma_{h_H}^2\right)\left(\left(\beta_L + E(h_L)\right)^2 + \sigma_{h_L}^2\right)} \qquad (24.3)$$

By moving some terms from the right-hand side to the left-hand side we get:

$$4 - \left(\left(\beta_H + E(h_H)\right)^2 + \sigma_{h_H}^2\right) + \left(\left(\beta_L + E(h_L)\right)^2 + \sigma_{h_L}^2\right)$$

$$= 2\rho_{PL,PH}\sqrt{\left(\left(\beta_H + E(h_H)\right)^2 + \sigma_{h_H}^2\right)\left(\left(\beta_L + E(h_L)\right)^2 + \sigma_{h_L}^2\right)}$$

If we divide both sides by everything remaining on the right-hand side except for correlation:

$$\rho_{PL,PH} = \frac{4 - \left(\left(\left(\beta_H + E(h_H)\right)^2 + \sigma_{h_H}^2\right) + \left(\left(\beta_L + E(h_L)\right)^2 + \sigma_{h_L}^2\right)\right)}{2\sqrt{\left(\left(\beta_H + E(h_H)\right)^2 + \sigma_{h_H}^2\right)\left(\left(\beta_L + E(h_L)\right)^2 + \sigma_{h_L}^2\right)}}$$

This was an intermediate step in video 24. Then I skipped to the conclusion.

Let's pause for a moment and go back to identity (24.3):

$$4 = \left(\left(\beta_H + E(h_H)\right)^2 + \sigma_{h_H}^2\right) + \left(\left(\beta_L + E(h_L)\right)^2 + \sigma_{h_L}^2\right)$$

$$+2\rho_{PL,PH}\sqrt{\left(\left(\beta_H + E(h_H)\right)^2 + \sigma_{h_H}^2\right)\left(\left(\beta_L + E(h_L)\right)^2 + \sigma_{h_L}^2\right)} \qquad (24.3)$$

Since the sum of the low and high ideal betas equals 2, then the sum of the ideal betas squared equals 4.

$$(\beta_L + \beta_H)^2 = 4$$

Recall that the sum of the expected *h*s is zero. Then we can add zero to the above equation in the form expected low *h* plus expected high *h*.

$$E(h_L) + E(h_H) = 0 \qquad (22.5)$$

$$\left(\left(\beta_L + E(h_L)\right) + \left(\beta_H + E(h_H)\right)\right)^2 = 4$$

Expanding the terms in the square gives:

$$\left(\beta_L + E(h_L)\right)^2 + 2\left(\beta_L + E(h_L)\right)\left(\beta_H + E(h_H)\right) + \left(\beta_H + E(h_H)\right)^2 = 4 \tag{29.1}$$

Substituting (29.1) into the formula for correlation:

$$\rho_{PL,PH} = \frac{4 - \left(\left(\left(\beta_H + E(h_H)\right)^2 + \sigma_{h_H}^2\right) + \left(\left(\beta_L + E(h_L)\right)^2 + \sigma_{h_L}^2\right)\right)}{2\sqrt{\left(\left(\beta_H + E(h_H)\right)^2 + \sigma_{h_H}^2\right)\left(\left(\beta_L + E(h_L)\right)^2 + \sigma_{h_L}^2\right)}}$$

gives us:

$$\rho_{PL,PH} =$$

$$\frac{\left(\beta_L + E(h_L)\right)^2 + 2\left(\beta_L + E(h_L)\right)\left(\beta_H + E(h_H)\right) + \left(\beta_H + E(h_H)\right)^2 - \left(\left(\left(\beta_H + E(h_H)\right)^2 + \sigma_{h_H}^2\right) + \left(\left(\beta_L + E(h_L)\right)^2 + \sigma_{h_L}^2\right)\right)}{2\sqrt{\left(\left(\beta_H + E(h_H)\right)^2 + \sigma_{h_H}^2\right)\left(\left(\beta_L + E(h_L)\right)^2 + \sigma_{h_L}^2\right)}}$$

The parts in red cancel, and then canceling the 2s from the numerator and the denominator leaves us with (24.4):

$$\rho_{PL,PH} = \frac{\left(\beta_L + E(h_L)\right)\left(\beta_H + E(h_H)\right) - \sigma_{h_H}^2}{\sqrt{\left(\left(\beta_H + E(h_H)\right)^2 + \sigma_{h_H}^2\right)\left(\left(\beta_L + E(h_L)\right)^2 + \sigma_{h_L}^2\right)}} \tag{24.4}$$

If we go back to (29.1) and substitute it into (24.3), then we get another large mess:

$$\left(\beta_L + E(h_L)\right)^2 + 2\left(\beta_L + E(h_L)\right)\left(\beta_H + E(h_H)\right) + \left(\beta_H + E(h_H)\right)^2 =$$

$$\left(\left(\beta_H + E(h_H)\right)^2 + \sigma_{h_H}^2\right) + \left(\left(\beta_L + E(h_L)\right)^2 + \sigma_{h_L}^2\right) + 2\rho_{PL,PH}\sqrt{\left(\left(\beta_H + E(h_H)\right)^2 + \sigma_{h_H}^2\right)\left(\left(\beta_L + E(h_L)\right)^2 + \sigma_{h_L}^2\right)}$$

We can subtract the parts in red from both sides and get:

$$2\left(\beta_L + E(h_L)\right)\left(\beta_H + E(h_H)\right) = 2\sigma_{h_L}^2 + 2\rho_{PL,PH}\sqrt{\left(\left(\beta_H + E(h_H)\right)^2 + \sigma_{h_H}^2\right)\left(\left(\beta_L + E(h_L)\right)^2 + \sigma_{h_L}^2\right)}$$

Then divide both sides by 2:

$$\left(\beta_L + E(h_L)\right)\left(\beta_H + E(h_H)\right) = \sigma_{h_L}^2 + \rho_{PL,PH}\sqrt{\left(\left(\beta_H + E(h_H)\right)^2 + \sigma_{h_H}^2\right)\left(\left(\beta_L + E(h_L)\right)^2 + \sigma_{h_L}^2\right)} \tag{29.2}$$

(29.2) is an identity that holds regardless of which scenario or condition we are in.

To find out what it means will require some algebra and an example. First the algebra.

Recall that by definition, if correlation between perturbations equals 1, then the $\sigma_h^2 = 0$. Plugging these values into (29.2) gives us.

$$\left(\beta_L + E(h_L)\right)\left(\beta_H + E(h_H)\right) = 0 + 1\sqrt{\left(\left(\beta_H + E(h_H)\right)^2 + 0\right)\left(\left(\beta_L + E(h_L)\right)^2 + 0\right)}$$

$$= \sqrt{\left(\left(\beta_H + E(h_H)\right)^2\right)\left(\left(\beta_L + E(h_L)\right)^2\right)}$$

$$= \left(\beta_L + E(h_L)\right)\left(\beta_H + E(h_H)\right) \checkmark$$

The algebra works out when correlation equals 1.

Now for an involved example.

First, we need more algebra and then a prediction. Recall that in example 13.1, correlation between perturbations was set to zero. Which means (from 29.2)

$$\left(\beta_L + E(h_L)\right)\left(\beta_H + E(h_H)\right) = \sigma_{h_L}^2 + 0\sqrt{\left(\left(\beta_H + E(h_H)\right)^2 + \sigma_{h_H}^2\right)\left(\left(\beta_L + E(h_L)\right)^2 + \sigma_{h_L}^2\right)}$$

$$\left(\beta_L + E(h_L)\right)\left(\beta_H + E(h_H)\right) = \sigma_{h_L}^2$$

The prediction is that once we derive $E(h_H)$, we will know the σ_h^2. It will take some work to get there.

From example 13.1 we have the following parameters:

$\beta_L = 0.5$ $\quad\quad$ $\beta_H = 1.5$

$\sigma_{PM} = 8.49837\%$ $\quad\quad$ $\sigma_{PL} = 3.333\%$ $\quad\quad$ $\sigma_{PH} = 16.666\%$

And as a needed reference, here is (23.2a) and (23.2b):

$$\sigma_{PL}^2 = \sigma_{PM}^2\left(\left(\beta_L + E(h_L)\right)^2 + \sigma_{h_L}^2\right) \tag{23.2a}$$

$$\sigma_{PH}^2 = \sigma_{PM}^2\left(\left(\beta_H + E(h_H)\right)^2 + \sigma_{h_H}^2\right) \tag{23.2b}$$

These equations will be solved simultaneously to give us $E(h_H)$ and σ_h^2.

First, we plug in all known parameter values:

$$0.00111 = 0.00722222\left(\left(0.5 + E(h_L)\right)^2 + \sigma_{h_L}^2\right)$$
$$0.02777 = 0.00722222\left(\left(1.5 + E(h_H)\right)^2 + \sigma_{h_H}^2\right)$$

Divide both sides of both equations by 0.00722222, and multiply the terms within the parenthesis:

$$0.15384613 = 0.25 + 1E(h_L) + \left(E(h_L)\right)^2 + \sigma_{h_L}^2$$
$$3.84615388 = 2.25 + 3E(h_H) + \left(E(h_H)\right)^2 + \sigma_{h_H}^2$$

Move all the constant terms to the left-hand side:

$$-0.09615387 = 1E(h_L) + \left(E(h_L)\right)^2 + \sigma_{h_L}^2$$
$$1.59615388 = 3E(h_H) + \left(E(h_H)\right)^2 + \sigma_{h_H}^2$$

Since $E(h_L) = -E(h_H)$, then by subtracting the first equation from the second we get:

$$1.692307749 = 4E(h_H)$$

$$E(h_H) = \quad 0.423076937 \quad \rightarrow \quad \left(E(h_H)\right)^2 = 0.1789941$$

$$E(h_L) = -0.423076937 \quad \rightarrow \quad \left(E(h_L)\right)^2 = 0.1789941$$

We can solve for $\sigma_{h_L}^2 = \sigma_{h_H}^2$ by plugging $E(h_L)$ into equation (23.2a):

$$\sigma_{PL}^2 = \sigma_{PM}^2 \left(\left(\beta_L + E(h_L)\right)^2 + \sigma_{h_L}^2\right) \tag{23.2a}$$

$$0.001111 = 0.00722222\left((0.5 - 0.4230769)^2 + \sigma_{h_L}^2\right)$$

$$\sigma_{h_L}^2 = \sigma_{h_H}^2 = 0.147928.$$

Now we need to test this result by plugging $E(h_L)$ into (29.2):

$$\left(\beta_L + E(h_L)\right)\left(\beta_H + E(h_H)\right) = (0.5 - 0.42307)(1.5 + .42307) = 0.147928$$

$$\left(\beta_L + E(h_L)\right)\left(\beta_H + E(h_H)\right) = 0.147928 = \sigma_{h_L}^2 = \sigma_{h_H}^2 \quad \checkmark$$

Identity (29.2) checks algebraically and numerically.

The same technique will work for the backward model as well.

Using the estimates for $\hat{\beta}_L$ and $\hat{\beta}_H$ used in the construction of Table 21.2:

$$\hat{\beta}_L = 0.943391 \qquad\qquad \hat{\beta}_H = 1.056609$$

The estimated forms of equations (23.2a) and (23.2b) are below as equations (29.3a) and (29.3b) and can be solved simultaneously to give us Table 29.1, which is an extension of Table 21.2.

$$\hat{\sigma}_{PL}^2 = \hat{\sigma}_{PM}^2 \left(\left(\hat{\beta}_L + \bar{h}_L\right)^2 + \hat{\sigma}_{h_L}^2\right) \tag{29.3a}$$

$$\hat{\sigma}_{PH}^2 = \hat{\sigma}_{PM}^2 \left(\left(\hat{\beta}_H + \bar{h}_H\right)^2 + \hat{\sigma}_{h_H}^2\right) \tag{29.3b}$$

Table 29.1
Estimated h and Variance of h for Realized Market Data
January 2015 – December 2019

	$\hat{\sigma}_{PL,PH}$	$\hat{\sigma}^2_{PL}$	$\hat{\sigma}^2_{PH}$	$\hat{\rho}_{PL,PH}$	$\hat{\sigma}^2_{PM}$	$\bar{h}_H$	$\hat{\sigma}^2_{h_H}$
1	0.00118	0.00027	0.00520	1.00000	0.00196	0.57295	0.00000
2	0.00129	0.00037	0.00532	0.92417	0.00207	0.54221	0.01667
3	0.00140	0.00047	0.00545	0.88049	0.00218	0.51460	0.03003
4	0.00151	0.00056	0.00557	0.85324	0.00229	0.48967	0.04078
5	0.00162	0.00066	0.00569	0.83549	0.00240	0.46704	0.04947
6	0.00173	0.00076	0.00582	0.82368	0.00251	0.44641	0.05649
7	0.00185	0.00086	0.00594	0.81579	0.00262	0.42752	0.06217
8	0.00196	0.00096	0.00606	0.81059	0.00273	0.41017	0.06677
9	0.00207	0.00106	0.00619	0.80731	0.00285	0.39417	0.07047
10	0.00218	0.00116	0.00631	0.80541	0.00296	0.37938	0.07344
11	0.00229	0.00126	0.00644	0.80452	0.00307	0.36565	0.07580

In video 24 it was claimed that inequality (23.8) was algebraically equivalent to inequality (24.5):

$$\frac{\left(\left(\beta_H+E(h_H)\right)^2+\sigma^2_{h_H}\right)}{\left(\left(\beta_L+E(h_L)\right)^2+\sigma^2_{h_L}\right)}+\frac{2\rho_{PL,PH}\sqrt{\left(\left(\beta_H+E(h_H)\right)^2+\sigma^2_{h_H}\right)}}{\sqrt{\left(\left(\beta_L+E(h_L)\right)^2+\sigma^2_{h_L}\right)}}>\frac{\beta_H^2}{\beta_L^2}+\frac{2\beta_H}{\beta_L} \tag{23.8}$$

$$\leftrightarrow$$

$$\left(\beta_L+E(h_L)\right)^2+\sigma^2_{h_L}<\beta_L^2 \tag{24.5}$$

This is not obvious. The first step is to substitute correlation from (24.4) into (23.8). That is this:

$$\rho_{PL,PH}=\frac{\left(\beta_L+E(h_L)\right)\left(\beta_H+E(h_H)\right)-\sigma^2_{h_H}}{\sqrt{\left(\left(\beta_H+E(h_H)\right)^2+\sigma^2_{h_H}\right)\left(\left(\beta_L+E(h_L)\right)^2+\sigma^2_{h_L}\right)}} \tag{24.4}$$

into this:

$$\frac{\left(\left(\beta_H+E(h_H)\right)^2+\sigma^2_{h_H}\right)}{\left(\left(\beta_L+E(h_L)\right)^2+\sigma^2_{h_L}\right)}+\frac{2\rho_{PL,PH}\sqrt{\left(\left(\beta_H+E(h_H)\right)^2+\sigma^2_{h_H}\right)}}{\sqrt{\left(\left(\beta_L+E(h_L)\right)^2+\sigma^2_{h_L}\right)}}>\frac{\beta_H^2}{\beta_L^2}+\frac{2\beta_H}{\beta_L} \tag{23.8}$$

Which leads to:

$$\frac{\left(\left(\beta_H+E(h_H)\right)^2+\sigma^2_{h_H}\right)}{\left(\left(\beta_L+E(h_L)\right)^2+\sigma^2_{h_L}\right)}+\frac{2\left(\left(\beta_L+E(h_L)\right)\left(\beta_H+E(h_H)\right)-\sigma^2_{h_H}\right)\sqrt{\left(\left(\beta_H+E(h_H)\right)^2+\sigma^2_{h_H}\right)}}{\sqrt{\left(\left(\beta_H+E(h_H)\right)^2+\sigma^2_{h_H}\right)\left(\left(\beta_L+E(h_L)\right)^2+\sigma^2_{h_L}\right)}\sqrt{\left(\left(\beta_L+E(h_L)\right)^2+\sigma^2_{h_L}\right)}}>\frac{\beta_H^2}{\beta_L^2}+\frac{2\beta_H}{\beta_L}$$

The parts in red cancel, and the parts in blue multiply to remove the square root:

$$\frac{\left(\left(\beta_H + E(h_H)\right)^2 + \sigma_{h_H}^2\right)}{\left(\left(\beta_L + E(h_L)\right)^2 + \sigma_{h_L}^2\right)} + \frac{2\left(\left(\beta_L + E(h_L)\right)\left(\beta_H + E(h_H)\right) - \sigma_{h_H}^2\right)}{\left(\left(\beta_L + E(h_L)\right)^2 + \sigma_{h_L}^2\right)} > \frac{\beta_H^2}{\beta_L^2} + \frac{2\beta_H}{\beta_L}$$

The denominator of both terms on the left-hand side is the same, and so the numerators can be combined. Also, on the right-hand side, the numerators can be combined over a new common denominator.

$$\frac{\left(\left(\beta_H + E(h_H)\right)^2 + \sigma_{h_H}^2\right) + 2\left(\left(\beta_L + E(h_L)\right)\left(\beta_H + E(h_H)\right) - \sigma_{h_H}^2\right)}{\left(\left(\beta_L + E(h_L)\right)^2 + \sigma_{h_L}^2\right)} > \frac{\beta_H^2 + 2\beta_H\beta_L}{\beta_L^2}$$

Next, the σ_h^2 terms are combined on the left, and the β_H is factored out on the right:

$$\frac{\left(\left(\beta_H + E(h_H)\right)^2\right) + 2\left(\left(\beta_L + E(h_L)\right)\left(\beta_H + E(h_H)\right)\right) - \sigma_{h_H}^2}{\left(\left(\beta_L + E(h_L)\right)^2 + \sigma_{h_L}^2\right)} > \frac{(\beta_H)(\beta_H + 2\beta_L)}{\beta_L^2}$$

The left-hand side gets factored and the right-hand side gets rearranged:

$$\frac{\left(\beta_H + E(h_H)\right)\left(\left(\beta_H + E(h_H)\right) + 2\left(\beta_L + E(h_L)\right)\right) - \sigma_{h_H}^2}{\left(\left(\beta_L + E(h_L)\right)^2 + \sigma_{h_L}^2\right)} > \frac{(\beta_H)(2 + \beta_L)}{\beta_L^2}$$

Since $E(h_L) = -E(h_H)$, the left-hand side can be combined and rearranged, and the right-hand side gets rearranged as well:

$$\frac{\left(\beta_H + E(h_H)\right)\left(2 + \left(\beta_L + E(h_L)\right)\right) - \sigma_{h_H}^2}{\left(\left(\beta_L + E(h_L)\right)^2 + \sigma_{h_L}^2\right)} > \frac{(2 - \beta_L)(2 + \beta_L)}{\beta_L^2}$$

Again, since $E(h_L) = -E(h_H)$, the left-hand side can be rearranged, and the right-hand side becomes the difference of two squares:

$$\frac{\left(2 - \left(\beta_L + E(h_L)\right)\right)\left(2 + \left(\beta_L + E(h_L)\right)\right) - \sigma_{h_H}^2}{\left(\left(\beta_L + E(h_L)\right)^2 + \sigma_{h_L}^2\right)} > \frac{4 - \beta_L^2}{\beta_L^2}$$

Multiplying out the numerator of the left-hand side gives us the difference of two squares plus an extra term:

$$\frac{4 - \left(\beta_L + E(h_L)\right)^2 - \sigma_{h_H}^2}{\left(\left(\beta_L + E(h_L)\right)^2 + \sigma_{h_L}^2\right)} > \frac{4 - \beta_L^2}{\beta_L^2}$$

If we multiply through by the denominator of both sides we get:

$$4\beta_L^2 - \beta_L^2\left(\beta_L + E(h_L)\right)^2 - \beta_L^2\sigma_{h_H}^2 > 4\left(\beta_L + E(h_L)\right)^2 + 4\sigma_{h_L}^2 - \beta_L^2\left(\beta_L + E(h_L)\right)^2 - \beta_L^2\sigma_{h_L}^2$$

The parts in red can be subtracted from both sides, then we divide through by four:

$$4\beta_L^2 > 4\left(\beta_L + E(h_L)\right)^2 + 4\sigma_{h_L}^2 \quad \rightarrow$$

$$\beta_L^2 > \left(\beta_L + E(h_L)\right)^2 + \sigma_{h_L}^2 \tag{24.5}$$

Which gives us (24.5).

It has now been shown that (23.8) is equivalent to (24.5).

This concludes this video.

Alpha Found Video 30

Closing the Open Problem from Video 9, part 1

Welcome to the overdue part 30 of Alpha Found.

It's possible that everything in this video has been said before by others, but if so, I am unaware of it.

This video is off the main path from the other videos, but there is a problem leftover from video 9 in need of a solution. In the ninth video of this series, it was proved that if an index is broken into two equal-weight halves, the correlation of the high CAPM beta portfolio is necessarily higher than the correlation of the low CAPM beta portfolio.

$$\hat{\rho}'_{L,M} < \hat{\rho}'_{H,M} \tag{9.8}$$

Then it was observed that in at least two studies (and very likely many others) that the correlation of each high CAPM beta decile was higher than the correlation of the symmetric low CAPM beta decile, as seen here:

Table 9.2
Correlations Derived from *The Volatiltiy Effect: Lower Risk Without Lower Return* by David Blitz and Pim Van Vliet (2007)

US Market Correlation Statistics Compared by Matched CAPM Beta Deciles 1986 - 2006

Matched Correlation Deciles	Correlations	Difference	
6th vs 5th	0.9728 - 0.9455 =	0.02723	In 5 of 5 cases the correlation of the
7th vs 4th	0.9595 - 0.9249 =	0.03459	higher β' decile is higher than the
8th vs 3rd	0.9583 - 0.9105 =	0.04774	correlation of the symmetric decile.
9th vs 2nd	0.9213 - 0.8737 =	0.04756	
10th vs 1st	0.8292 - 0.6413 =	0.18798	

European Market Correlation Statistics Compared by Matched CAPM Beta Deciles 1986 - 2006

Matched Correlation Deciles	Correlations	Difference	
6th vs 5th	0.9635 - 0.9564 =	0.00709	In 5 of 5 cases the correlation of the
7th vs 4th	0.9763 - 0.9630 =	0.01330	higher β' decile is higher than the
8th vs 3rd	0.9703 - 0.9503 =	0.01997	correlation of the symmetric decile.
9th vs 2nd	0.9525 - 0.9120 =	0.04056	
10th vs 1st	0.9085 - 0.9032 =	0.00531	

Japanese Market Correlation Statistics Compared by Matched CAPM Beta Deciles 1986 - 2006

Matched Correlation Deciles	Correlations	Difference	
6th vs 5th	0.9834 - 0.9800 =	0.00341	In 5 of 5 cases the correlation of the
7th vs 4th	0.9823 - 0.9734 =	0.00894	higher β' decile is higher than the
8th vs 3rd	0.9773 - 0.9543 =	0.02294	correlation of the symmetric decile.
9th vs 2nd	0.9600 - 0.9317 =	0.02830	
10th vs 1st	0.9252 - 0.8628 =	0.06232	

And here:

Table 9.4
Correlations Derived from *The Capital Asset Pricing Model: Some Empirical Tests*
by Fischer Black, Michael C. Jensen, and Myron Scholes (1972) aka BJS

BJS Correlation Statistics Compared by Matched Beta Deciles 1931 - 1965

Matched Correlation Deciles	Correlations	Difference	
6th vs 5th	0.9915 - 0.9833 =	0.00820	In 5 of 5 cases the correlation of the higher β decile is
7th vs 4th	0.9914 - 0.9851 =	0.00630	higher than the correlation of the symmetric decile.
8th vs 3rd	0.9882 - 0.9793 =	0.00890	
9th vs 2nd	0.9875 - 0.9560 =	0.03150	
10th vs 1st	0.9625 - 0.8981 =	0.06440	

BJS Correlation Statistics Compared by Matched Beta Deciles Subperiod 1

Matched Correlation Deciles	Correlations	Difference	
6th vs 5th	0.9948 - 0.9835 =	0.01123	In 5 of 5 cases the correlation of the higher β decile is
7th vs 4th	0.9940 - 0.9876 =	0.00644	higher than the correlation of the symmetric decile.
8th vs 3rd	0.9900 - 0.9842 =	0.00584	
9th vs 2nd	0.9901 - 0.9643 =	0.02576	
10th vs 1st	0.9042 - 0.9042 =	0.07283	

BJS Correlation Statistics Compared by Matched Beta Deciles Subperiod 2

Matched Correlation Deciles	Correlations	Difference	
6th vs 5th	0.9791 - 0.9854 =	-0.00629	In 4 of 5 cases the correlation of the higher β decile is
7th vs 4th	0.9822 - 0.9756 =	0.00666	higher than the correlation of the symmetric decile.
8th vs 3rd	0.9828 - 0.9695 =	0.01329	It would not be surprising to find the reported β or
9th vs 2nd	0.9791 - 0.9405 =	0.03858	standard deviation of decile 5 or 6 is an error, given
10th vs 1st	0.9019 - 0.8956 =	0.00637	the difference in correlations is negative.

BJS Correlation Statistics Compared by Matched Beta Deciles Subperiod 3

Matched Correlation Deciles	Correlations	Difference	
6th vs 5th	0.9875 - 0.9824 =	0.00515	In 5 of 5 cases the correlation of the higher β decile is
7th vs 4th	0.9858 - 0.9801 =	0.00571	higher than the correlation of the symmetric decile.
8th vs 3rd	0.9843 - 0.9689 =	0.01534	
9th vs 2nd	0.9774 - 0.9394 =	0.03809	
10th vs 1st	0.9639 - 0.8705 =	0.09337	

BJS Correlation Statistics Compared by Matched Beta Deciles Subperiod 4

Matched Correlation Deciles	Correlations	Difference	
6th vs 5th	0.9915 - 0.9833 =	0.00820	In 5 of 5 cases the correlation of the higher β decile is
7th vs 4th	0.9914 - 0.9851 =	0.00630	higher than the correlation of the symmetric decile.
8th vs 3rd	0.9882 - 0.9793 =	0.00890	
9th vs 2nd	0.9875 - 0.9560 =	0.03150	
10th vs 1st	0.9625 - 0.8981 =	0.06440	

There was one exception in the second time period of the paper by Black, Jensen, and Scholes, for the fifth and sixth deciles.

In this and the next video we pursue the relationship between the correlations of the CAPM beta deciles further.

It turns out the observed relationship is not just a happy coincidence. It will likely exist for symmetric deciles as it does for halves. In retrospect, it should have been obvious.

The mathematics will be a little more formal than in prior videos, so this is a half-step up in sophistication.

Recall equation (9.9):

$$\hat{\rho}'_{i,M} = \sqrt{\frac{\sum_{t=1}^{T}\left((R'_{Mt}-r_f)-(\bar{R}'_M-r_f)\right)^2}{\sum_{t=1}^{T}\left((R'_{Mt}-r_f)+\frac{\hat{\varepsilon}'_{i,t}}{\hat{\beta}'_i}-(\bar{R}'_M-r_f)\right)^2}} \quad (9.9)$$

This could have been simplified further, which I didn't do originally because I thought the equation as presented was more intuitive than the simpler alternative. To simplify further, the risk-free rate of return is a constant that is both added in and subtracted out and so it drops out altogether, leaving us with line (30.1):

$$\hat{\rho}'_{i,M} = \sqrt{\frac{\sum_{t=1}^{T}(R'_{Mt}-\bar{R}'_M)^2}{\sum_{t=1}^{T}\left(R'_{Mt}+\frac{\hat{\varepsilon}'_{i,t}}{\hat{\beta}'_i}-\bar{R}'_M\right)^2}} \quad (30.1)$$

Where the subscript *i* tells us which security we are talking about, and the subscript *t* tells us what time period we are talking about.

If we expand the denominator we get a mess:

$$\sum_{t=1}^{T}\left(R'_{Mt}+\frac{\hat{\varepsilon}'_{i,t}}{\hat{\beta}'_i}-\bar{R}'_M\right)^2 = \sum_{t=1}^{T}\left(R'^2_{Mt}+\left(\frac{\hat{\varepsilon}'_{i,t}}{\hat{\beta}'_i}\right)^2+\bar{R}'^2_{Mt}+2R'_{Mt}\frac{\hat{\varepsilon}'_{i,t}}{\hat{\beta}'_i}-2\frac{\hat{\varepsilon}'_{i,t}}{\hat{\beta}'_i}\bar{R}'_M-2R'_{Mt}\bar{R}'_M\right)$$

Which when rearranged results in a smaller mess:

$$= \sum_{t=1}^{T}\left(R'^2_{Mt}+\left(\frac{\hat{\varepsilon}'_{i,t}}{\hat{\beta}'_i}\right)^2+\bar{R}'^2_{Mt}+2\frac{\hat{\varepsilon}'_{i,t}}{\hat{\beta}'_i}(R'_{Mt}-\bar{R}'_M)-2R'_{Mt}\bar{R}'_M\right)$$

When summed, the part in red will sum to zero and drop out, and if the terms are rearranged, again, we get a still smaller mess:

$$= \sum_{t=1}^{T}\left(R'^2_{Mt}-2R'_{Mt}\bar{R}'_M+\bar{R}'^2_{Mt}+\left(\frac{\hat{\varepsilon}'_{i,t}}{\hat{\beta}'_i}\right)^2\right)$$

The first three terms from the left can be simplified further, and the summation can be distributed to give:

$$= \sum_{t=1}^{T}(R'_{Mt}-\bar{R}'_M)^2+\sum_{t=1}^{T}\left(\frac{\hat{\varepsilon}'_{i,t}}{\hat{\beta}'_i}\right)^2$$

When this new arrangement is placed back into line (30.1) we get:

$$\hat{\rho}'_{i,M} = \sqrt{\frac{\sum_{t=1}^{T}\left(R'_{Mt}-\bar{R}'_{M}\right)^2}{\sum_{t=1}^{T}\left(R'_{Mt}-\bar{R}'_{M}\right)^2+\sum_{t=1}^{T}\left(\frac{\hat{\varepsilon}'_{i,t}}{\hat{\beta}'_{i}}\right)^2}} \tag{30.2}$$

Notice that the first summation on the left of the denominator is identical to the numerator, and will have the same value in any correlation calculation for every security or portfolio contained within the index over the same time period. Therefore, the difference in correlations between securities or portfolios comes down to the difference between the value of the right-hand summation in the denominator:

$$\sum_{t=1}^{T}\left(\frac{\hat{\varepsilon}'_{i,t}}{\hat{\beta}'_{i}}\right)^2 \tag{30.3}$$

The larger the value of (30.3), the lower the correlation.

With this, we have sufficient information to explain the pattern of differences in correlations of the symmetric deciles. However, a better foundation for the explanation will lead to a deeper understanding, so I am going to take the better foundation route.

In regression, the sum of the residuals across all time periods for each portfolio or security is 0, as shown:

$$\sum_{t=1}^{T}\hat{\varepsilon}'_{i,t} = 0 = \bar{\varepsilon}_i \tag{30.4}$$

Also, in regression, when the independent variable is a stock index, as in the context of CAPM, with alpha, there is an additional sum of residuals that sums to 0.

$$\sum_{i=1}^{n} w_i\hat{\varepsilon}'_{i,t} = 0 \tag{30.5}$$

Note here that the index of the summation is *i* not *t*.

That is, the weighted sum of the residuals in any given time period across all securities or all mutually exclusive and exhaustive portfolios comprising an index, is zero. This was said verbally and less formally in the discussion of line (9.2):

$$\hat{\sigma}'^2_{\varepsilon H} = \hat{\sigma}'^2_{\varepsilon L} = \hat{\sigma}'^2_{\varepsilon} \tag{9.2}$$

In the two portfolio market that has been used throughout these videos, each portfolio is equally weighted at 50% each.

Therefore, in any given time period *t*, the sum of the two residuals is zero:

$$\hat{\varepsilon}'_{L,t} + \hat{\varepsilon}'_{H,t} = 0$$

$$\hat{\varepsilon}'_{L,t} = -\hat{\varepsilon}'_{H,t} \tag{30.6}$$

We can take the absolute value of both sides of equation 30.6:

$$\left|\hat{\varepsilon}'_{L,t}\right| = \left|\hat{\varepsilon}'_{H,t}\right| \tag{30.7}$$

The equation in line (30.7) should be easier to work with than line (30.6), as long as we remember that the residuals of each of the two portfolios in each time period have opposite signs.

Since the magnitude of the residuals varies from one time period to next, but in each time period the magnitude of the residual is the same for each of the two portfolios, then the idiosyncratic risk is the same for each of the two portfolios:

$$\hat{\sigma}_{\varepsilon H}'^2 = \hat{\sigma}_{\varepsilon L}'^2 = \hat{\sigma}_{\varepsilon}'^2 \tag{9.2}$$

Where idiosyncratic risk is the sum of the squared residuals divided by their degrees of freedom:

$$\hat{\sigma}_{\varepsilon H}'^2 = \hat{\sigma}_{\varepsilon L}'^2 = \frac{\sum_{t=1}^{T}\left(\hat{\varepsilon}_{i,t}' - \bar{\varepsilon}_i'\right)^2}{n-2} = \frac{\sum_{t=1}^{T}\left(\hat{\varepsilon}_{i,t}'\right)^2}{n-2} \tag{30.8}$$

In this case, *i* equals L or H.

Since the absolute value of the residuals is equal in each time period:

$$\left|\hat{\varepsilon}_{L,t}'\right| = \left|\hat{\varepsilon}_{H,t}'\right| \tag{30.7}$$

And since by definition the low CAPM beta is less than the high CAPM beta:

$$\hat{\beta}_L' < \hat{\beta}_H' \tag{30.9}$$

then the reciprocal of the low CAPM beta is greater than the reciprocal of the high CAPM beta:

$$\frac{1}{\hat{\beta}_L'} > \frac{1}{\hat{\beta}_H'} \tag{30.10}$$

Therefore, the absolute value of the residuals divided by their respective CAPM beta have this relationship:

$$\frac{\left|\hat{\varepsilon}_{L,t}'\right|}{\hat{\beta}_L'} > \frac{\left|\hat{\varepsilon}_{H,t}'\right|}{\hat{\beta}_H'} \tag{30.11}$$

This is true for all *t*.

Which tells us immediately that the sum of squared residuals divided by their respective CAPM beta has the relationship found here, in line (30.12):

$$\sum_{t=1}^{T}\left(\frac{\hat{\varepsilon}_{L,t}'}{\hat{\beta}_L'}\right)^2 > \sum_{t=1}^{T}\left(\frac{\hat{\varepsilon}_{H,t}'}{\hat{\beta}_H'}\right)^2 \tag{30.12}$$

Plugging each piece of (30.12) into the denominator of (30.2), (the formula for correlation), gives us:

$$\sqrt{\frac{\sum_{t=1}^{T}\left(R_{Mt}' - \bar{R}_M'\right)^2}{\sum_{t=1}^{T}\left(R_{Mt}' - \bar{R}_M'\right)^2 + \sum_{t=1}^{T}\left(\frac{\hat{\varepsilon}_{L,t}'}{\hat{\beta}_L'}\right)^2}} < \sqrt{\frac{\sum_{t=1}^{T}\left(R_{Mt}' - \bar{R}_M'\right)^2}{\sum_{t=1}^{T}\left(R_{Mt}' - \bar{R}_M'\right)^2 + \sum_{t=1}^{T}\left(\frac{\hat{\varepsilon}_{H,t}'}{\hat{\beta}_H'}\right)^2}} \tag{30.13}$$

Which means the correlation of the low CAPM portfolio is less than the correlation of the high CAPM beta portfolio:

$$\hat{\rho}'_{L,M} < \hat{\rho}'_{H,M} \tag{9.8}$$

The conclusion in (9.8) was derived more simply in the 9th video.

Let's look more closely at line (30.7):

$$\left|\hat{\varepsilon}'_{L,t}\right| = \left|\hat{\varepsilon}'_{H,t}\right| \tag{30.7}$$

which when expanded looks like this:

$$\left|\hat{\varepsilon}'_{1,t} + \hat{\varepsilon}'_{2,t} + \hat{\varepsilon}'_{3,t} + \hat{\varepsilon}'_{4,t} + \hat{\varepsilon}'_{5,t}\right| = \left|\hat{\varepsilon}'_{6,t} + \hat{\varepsilon}'_{7,t} + \hat{\varepsilon}'_{8,t} + \hat{\varepsilon}'_{9,t} + \hat{\varepsilon}'_{10,t}\right| \tag{30.14}$$

And inequality (30.11):

$$\frac{\left|\hat{\varepsilon}'_{L,t}\right|}{\hat{\beta}'_L} > \frac{\left|\hat{\varepsilon}'_{H,t}\right|}{\hat{\beta}'_H} \tag{30.11}$$

can be expanded also:

$$\frac{\left|\hat{\varepsilon}'_{1,t}+\hat{\varepsilon}'_{2,t}+\hat{\varepsilon}'_{3,t}+\hat{\varepsilon}'_{4,t}+\hat{\varepsilon}'_{5,t}\right|}{\hat{\beta}'_1+\hat{\beta}'_2+\hat{\beta}'_3+\hat{\beta}'_4+\hat{\beta}'_5} > \frac{\left|\hat{\varepsilon}'_{6,t}+\hat{\varepsilon}'_{7,t}+\hat{\varepsilon}'_{8,t}+\hat{\varepsilon}'_{9,t}+\hat{\varepsilon}'_{10,t}\right|}{\hat{\beta}'_6+\hat{\beta}'_7+\hat{\beta}'_8+\hat{\beta}'_9+\hat{\beta}'_{10}} \tag{30.15}$$

Both sides of (30.14), as well as the numerator and denominator of both sides of (30.15) should be divided by 5 to account for the equal weight contribution of the pieces.

However, those fives immediately cancel, leaving us with what is on the screen.

The constituent parts of line (30.14) have some freedom to vary.

That is, the symmetric component residuals of line (30.14) do not have to have equal magnitude as their sum does, although they could have equal magnitude:

$$\left|\hat{\varepsilon}'_{1,t}\right| \neq \left|\hat{\varepsilon}'_{10,t}\right|, \;\; \left|\hat{\varepsilon}'_{2,t}\right| \neq \left|\hat{\varepsilon}'_{9,t}\right|, \ldots\;, \left|\hat{\varepsilon}'_{5,t}\right| \neq \left|\hat{\varepsilon}'_{6,t}\right|$$

More generally this can be written as:

$$\left|\hat{\varepsilon}'_{i,t}\right| \neq \left|\hat{\varepsilon}'_{10\;\;(i-1),t}\right| \qquad \text{for } 1 \leq i \leq 5 \tag{30.16}$$

What happens if both sides of line (30.16) are divided by their respective CAPM beta?

$$\frac{\left|\hat{\varepsilon}'_{i,t}\right|}{\hat{\beta}'_i} \neq \frac{\left|\hat{\varepsilon}'_{10\;\;(i-1),t}\right|}{\hat{\beta}'_{10\;\;(i-1)}} \qquad \text{for } 1 \leq i \leq 5 \tag{30.17}$$

A first glance it might not appear that we have gained anything, yet we have.

Since by definition the betas have this order:

$$\hat{\beta}'_1 < \hat{\beta}'_2 < \cdots < \hat{\beta}'_9 < \hat{\beta}'_{10} \tag{30.18}$$

then the reciprocals have this order:

$$\frac{1}{\hat{\beta}'_1} > \frac{1}{\hat{\beta}'_2} > \cdots > \frac{1}{\hat{\beta}'_9} > \frac{1}{\hat{\beta}'_{10}} \tag{30.19}$$

For a CAPM beta less than 1, the absolute value of the residual divided by beta is greater than the residual itself:

$$\frac{\left|\hat{\varepsilon}'_{i,t}\right|}{\hat{\beta}'_i} > \left|\hat{\varepsilon}'_{i,t}\right| \qquad \text{for } 1 \le i \le 5 \tag{30.20a}$$

Furthermore, for a CAPM beta greater than 1, the absolute value of the residual divided by beta is less than the absolute value of the residual:

$$\frac{\left|\hat{\varepsilon}'_{10-(i-1),t}\right|}{\hat{\beta}'_{10\ \ (i-1)}} < \left|\hat{\varepsilon}'_{10\ \ (i-1),t}\right| \qquad \text{for } 1 \le i \le 5 \tag{30.20b}$$

In decile portfolios, it’s safe the assume that there is a reasonable amount of diversification, though not complete diversification.

With this in mind, in light of line (30.14):

$$\left|\hat{\varepsilon}'_{1,t} + \hat{\varepsilon}'_{2,t} + \hat{\varepsilon}'_{3,t} + \hat{\varepsilon}'_{4,t} + \hat{\varepsilon}'_{5,t}\right| = \left|\hat{\varepsilon}'_{6,t} + \hat{\varepsilon}'_{7,t} + \hat{\varepsilon}'_{8,t} + \hat{\varepsilon}'_{9,t} + \hat{\varepsilon}'_{10,t}\right| \tag{30.14}$$

and in light of lines (30.20a) - (30.20b), there is a tendency for the relationship below to hold:

$$\frac{\left|\hat{\varepsilon}'_{i,t}\right|}{\hat{\beta}'_i} > \frac{\left|\hat{\varepsilon}'_{10\ \ (i-1),t}\right|}{\hat{\beta}'_{10-(i-1)}} \qquad \text{for } 1 \le i \le 5 \tag{30.21}$$

The magnitude of the residuals of the symmetric deciles are roughly equal in each time period. There is no obvious reason to think that any one residual magnitude of line (30.14) is greater than or less than its symmetric residual magnitude, except in the 1st and 10th deciles.

This will be discussed in the next video.

Dividing the residual of the lower CAPM beta decile by a number less than 1 (the lower CAPM beta), and dividing the residual of the higher CAPM beta decile by a number greater than one (the higher CAPM beta) leads to the left-hand side of (30.21) being more likely than not to exceed the right-hand side. The result is a tendency, not a certainty, for each pair of observations. (By the way, it is possible that the 6th decile has a CAPM beta less than 1.)

Since for individual pairs of observations this is more likely than not to be true, then it is even more likely to be true for the sum of squared observations across time.

That is, there is a strong tendency for the sum of squared residuals divided by their low CAPM beta squared to exceed the sum of squared residuals divided by the high CAPM beta squared:

$$\sum_{t=1}^{T} \left(\frac{\hat{\varepsilon}'_{i,t}}{\hat{\beta}'_i}\right)^2 > \sum_{t=1}^{T} \left(\frac{\hat{\varepsilon}'_{10\ \ (i-1),t}}{\hat{\beta}'_{10\ \ (i-1)}}\right)^2 \qquad \text{for } 1 \le i \le 5 \tag{30.22}$$

(compare to (30.12))

If both sides of (30.22) are divided (n-2):

$$\left(\frac{1}{\hat{\beta}_i'}\right)^2 \sum_{t=1}^{T} \frac{\left(\hat{\varepsilon}_{i,t}'\right)^2}{n-2} > \left(\frac{1}{\hat{\beta}_{10-(i-1)}'}\right)^2 \sum_{t=1}^{T} \frac{\left(\hat{\varepsilon}_{10\ \ (i-1),t}'\right)^2}{n-2} \qquad \text{for } 1 \le i \le 5 \qquad (30.23)$$

Then by line (30.8) we get:

$$\frac{\hat{\sigma}_{\varepsilon(i)}'^2}{\hat{\beta}_i'^2} > \frac{\hat{\sigma}_{\varepsilon(10-(i-1))}'^2}{\hat{\beta}_{10-(i-1)}'^2} \qquad \text{for } 1 \le i \le 5 \qquad (30.24)$$

Or equivalently by multiplying both sides by the CAPM beta on the right-hand side:

$$\left(\frac{\hat{\beta}_{10-(i-1)}'}{\hat{\beta}_i'}\right)^2 \hat{\sigma}_{\varepsilon(i)}'^2 > \hat{\sigma}_{\varepsilon(10\ \ (i-1))}'^2$$

(30.25)

By construction, the tendency for (30.22) - (30.25) to be true is made stronger or weaker according to this ordering:

$$\frac{\hat{\beta}_{10}'}{\hat{\beta}_1'} > \frac{\hat{\beta}_9'}{\hat{\beta}_2'} > \frac{\hat{\beta}_8'}{\hat{\beta}_3'} > \frac{\hat{\beta}_7'}{\hat{\beta}_4'} > \frac{\hat{\beta}_6'}{\hat{\beta}_5'} > 1 \qquad (30.26)$$

That is, the further the symmetric decile pairs are from the center, the more likely the inequalities of lines (30.22) - (30.25) are satisfied, subject to the constraints of lines (30.14) and (30.15) in each time period:

$$\left|\hat{\varepsilon}_{1,t}' + \hat{\varepsilon}_{2,t}' + \hat{\varepsilon}_{3,t}' + \hat{\varepsilon}_{4,t}' + \hat{\varepsilon}_{5,t}'\right| = \left|\hat{\varepsilon}_{6,t}' + \hat{\varepsilon}_{7,t}' + \hat{\varepsilon}_{8,t}' + \hat{\varepsilon}_{9,t}' + \hat{\varepsilon}_{10,t}'\right| \qquad (30.14)$$

$$\frac{\left|\hat{\varepsilon}_{1,t}'+\hat{\varepsilon}_{2,t}'+\hat{\varepsilon}_{3,t}'+\hat{\varepsilon}_{4,t}'+\hat{\varepsilon}_{5,t}'\right|}{\hat{\beta}_1'+\hat{\beta}_2'+\hat{\beta}_3'+\hat{\beta}_4'+\hat{\beta}_5'} > \frac{\left|\hat{\varepsilon}_{6,t}'+\hat{\varepsilon}_{7,t}'+\hat{\varepsilon}_{8,t}'+\hat{\varepsilon}_{9,t}'+\hat{\varepsilon}_{10,t}'\right|}{\hat{\beta}_6'+\hat{\beta}_7'+\hat{\beta}_8'+\hat{\beta}_9'+\hat{\beta}_{10}'} \qquad (30.15)$$

To summarize, there is a strong tendency for the correlation of the low CAPM beta decile to be lower than the correlation of the symmetric high CAPM beta decile.

$$\hat{\rho}_{i,M}' < \hat{\rho}_{10-(i-1),M}' \qquad \text{for } 1 \le i \le 5 \qquad (30.27)$$

I want to end this video with a useful note on equation (30.2):

$$\hat{\rho}_{i,M}' = \sqrt{\frac{\Sigma\left(R_{Mt}'-\bar{R}_M'\right)^2}{\Sigma\left(R_{Mt}'-\bar{R}_M'\right)^2+\Sigma\left(\frac{\hat{\varepsilon}_{i,t}'}{\hat{\beta}_i'}\right)^2}} \qquad (30.2)$$

Recall equation (9.10):

$$\hat{\rho}_{i,M}' = \frac{\hat{\beta}_i'\hat{\sigma}_{i,M}'}{\hat{\sigma}_i'} \qquad (9.10)$$

If we square both sides:

$$\hat{\rho}'^2_{i,M} = \frac{\hat{\beta}'^2_i \hat{\sigma}'^2_M}{\hat{\sigma}'^2_i}$$

And then expand the denominator of the right-hand side we get:

$$= \frac{\hat{\beta}'^2_i \hat{\sigma}'^2_M}{\hat{\beta}'^2_i \hat{\sigma}'^2_M + \hat{\sigma}'^2_{\varepsilon(i)}}$$

Then canceling CAPM beta gives:

$$= \frac{\hat{\sigma}'^2_M}{\hat{\sigma}'^2_M + \frac{\hat{\sigma}'^2_{\varepsilon(i)}}{\hat{\beta}'^2_i}}$$

By taking the square root of both sides, we get line (30.28):

$$\hat{\rho}'_{i,M} = \sqrt{\frac{\hat{\sigma}'^2_M}{\hat{\sigma}'^2_M + \frac{\hat{\sigma}'^2_{\varepsilon(i)}}{\hat{\beta}'^2_i}}} \qquad (30.28)$$

More fully, (30.2) is equivalent to (30.28):

$$\hat{\rho}'_{i,M} = \sqrt{\frac{\Sigma\left(R'_{Mt} - \bar{R}'_M\right)^2}{\Sigma\left(R'_{Mt} - \bar{R}'_M\right)^2 + \Sigma\left(\frac{\hat{\varepsilon}'_t}{\hat{\beta}'}\right)^2}} = \sqrt{\frac{\hat{\sigma}'^2_M}{\hat{\sigma}'^2_M + \frac{\hat{\sigma}'^2_{\varepsilon(i)}}{\hat{\beta}'^2_i}}} \qquad (30.29)$$

In the next video these formulas and others will be applied to the data from Blitz and Van Vliet, and also from Black, Jensen, and Scholes.

Then we move forward from analyzing symmetric deciles to analyzing consecutive deciles.

Alpha Found Video 31

Closing the Open Problem from Video 9, part 2

Welcome once more to Alpha Found part 31.

In this video, the formulas of the last video are applied to the data from Blitz and Van Vliet, and also to the data from Black, Jensen, and Scholes. Which means there are several tables of statistics.

After we finish analyzing the relationship between correlations of symmetric deciles, we turn our attention to consecutive deciles.

To begin, we need idiosyncratic risk for each decile for each grouping, from each of the two studies.

Since variance of a decile equals CAPM beta squared times market variance plus idiosyncratic variance:

$$\hat{\sigma}_i'^2 = \hat{\beta}_i'^2 \hat{\sigma}_M'^2 + \hat{\sigma}_{\varepsilon(i)}'^2 \tag{5.4}$$

Then idiosyncratic risk is calculated in this manner:

$$\hat{\sigma}_{\varepsilon(i)}'^2 = \hat{\sigma}_i'^2 - \hat{\beta}_i'^2 \hat{\sigma}_M'^2 \tag{31.1}$$

Using line (31.1), here in Table 31.1 are the estimates of idiosyncratic risk for Blitz and Van Vliet, and for Black, Jensen, and Scholes.

Table 31.1
Idiosyncratic Risk by Decile (31.1)
from *The Volatility Effect: Lower Risk Without Lower Return*
by David Blitz and Pim Van Vliet (2007)

	1	2	3	4	5	6	7	8	9	10
US Market	0.00848	0.00444	0.00405	0.00365	0.00306	0.00150	0.00257	0.00350	0.01007	0.04162
European Market	0.00283	0.00339	0.00221	0.00207	0.00252	0.00227	0.00169	0.00239	0.00521	0.01438
Japanese Market	0.00590	0.00428	0.00343	0.00212	0.00183	0.00164	0.00189	0.00288	0.00576	0.01569

Statistics from *The Capital Asset Pricing Model: Some Empirical Tests*
by Fischer Black, Michael C. Jensen, and Myron Scholes (1972)

	1	2	3	4	5	6	7	8	9	10
Subperiod 1	0.00132	0.00074	0.00046	0.00047	0.00072	0.00031	0.00042	0.00081	0.00100	0.00285
Subperiod 2	0.00030	0.00022	0.00015	0.00013	0.00010	0.00016	0.00017	0.00020	0.00029	0.00263
Subperiod 3	0.00010	0.00008	0.00005	0.00005	0.00005	0.00004	0.00005	0.00006	0.00011	0.00024
Subperiod 4	0.00012	0.00012	0.00007	0.00005	0.00006	0.00005	0.00007	0.00006	0.00010	0.00023

There is not much interesting to see in Table 31.1, except to note that the 1st and 10th deciles of each grouping tend to have higher idiosyncratic risk than the middle 8 deciles.

Recall from the last video that the difference in correlation between any two securities or portfolios comes down to the value of line (30.3):

$$\sum_{t=1}^{T}\left(\frac{\hat{\varepsilon}'_{i,t}}{\hat{\beta}'_i}\right)^2 \tag{30.3}$$

Where the larger the value of (30.3), the lower the correlation.

If we divide the values of Table 31.1 by the square of their respective CAPM betas, as seen in line (31.2):

$$\frac{\hat{\sigma}'^2_{\varepsilon(i)}}{\hat{\beta}'^2_i} = \frac{1}{n-2}\sum_{t=1}^{T}\left(\frac{\hat{\varepsilon}'_{i,t}}{\hat{\beta}'_i}\right)^2 \tag{31.2}$$

We get Table 31.2:

Table 31.2
Idiosyncratic Risk Divided by CAPM Beta Squared by Decile (31.2)
from *The Volatility Effect: Lower Risk Without Lower Return*
by David Blitz and Pim Van Vliet (2007)

	1	2	3	4	5	6	7	8	9	10
US Market	0.04187	0.00906	0.00603	0.00494	0.00347	0.00166	0.00252	0.00260	0.00521	0.01328
European Market	0.00691	0.00620	0.00328	0.00240	0.00286	0.00237	0.00150	0.00190	0.00313	0.00648
Japanese Market	0.01587	0.00703	0.00453	0.00256	0.00191	0.00157	0.00168	0.00218	0.00394	0.00778

Statistics from *The Capital Asset Pricing Model: Some Empirical Tests*
by Fischer Black, Michael C. Jensen, and Myron Scholes (1972)

	1	2	3	4	5	6	7	8	9	10
Subperiod 1	0.00562	0.00190	0.00082	0.00064	0.00085	0.00027	0.00030	0.00051	0.00051	0.00120
Subperiod 2	0.00096	0.00051	0.00025	0.00020	0.00012	0.00017	0.00014	0.00014	0.00017	0.00089
Subperiod 3	0.00042	0.00018	0.00009	0.00005	0.00005	0.00003	0.00004	0.00004	0.00006	0.00010
Subperiod 4	0.00032	0.00026	0.00011	0.00007	0.00007	0.00005	0.00006	0.00004	0.00006	0.00011

Visual inspection will confirm that each entry follows the pattern found in line (30.24), with the one exception in the 5th and 6th deciles of the second time period of BJS:

$$\frac{\hat{\sigma}'^2_{\varepsilon(i)}}{\hat{\beta}'^2_i} > \frac{\hat{\sigma}'^2_{\varepsilon(10\ \ (i-1))}}{\hat{\beta}'^2_{10\ \ (i-1)}} \tag{30.24}$$

Rather than perform a visual inspection, we can subtract out the right-hand side of line (30.24), which will net out the differences. The result is this inequality:

$$\left(\frac{\hat{\sigma}^2_{\varepsilon(i)}}{\hat{\beta}'_i}\right)^2 - \left(\frac{\hat{\sigma}'_{\varepsilon(10\ \ (i-1))}}{\hat{\beta}'_{10\ \ (i-1)}}\right)^2 > 0 \qquad \text{for } 1 \le i \le 5 \tag{31.3}$$

Line (31.3) is the mathematical foundation for the observed relationship between correlations of symmetric deciles.

Here is Table 31.3, with the results from line (31.3) for each grouping of each study.

Table 31.3
Differences in Idiosyncratic Risk Divided by CAPM Beta Squared by Symmetric Decile (31.3)
from *The Volatility Effect: Lower Risk Without Lower Return*
by David Blitz and Pim Van Vliet (2007)

	1st-10th	2nd-9th	3rd-8th	4th-7th	5th-6th
US Market	0.02859	0.00385	0.00181	0.00242	0.00181
European Market	0.00044	0.00307	0.00049	0.00089	0.00049
Japanese Market	0.00808	0.00309	0.00033	0.00088	0.00033

Statistics from *The Capital Asset Pricing Model: Some Empirical Tests*
by Fischer Black, Michael C. Jensen, and Myron Scholes (1972)

	1st-10th	2nd-9th	3rd-8th	4th-7th	5th-6th
Subperiod 1	0.00442	0.00139	0.00031	0.00033	0.00058
Subperiod 2	0.00007	0.00034	0.00011	0.00006	-0.00005
Subperiod 3	0.00032	0.00011	0.00004	0.00002	0.00001
Subperiod 4	0.00021	0.00020	0.00007	0.00001	0.00002

Line (31.3) holds for every entry of Table 31.3 but one, the 5^{th} and 6^{th} deciles of the second time period of BJS, the only entry with a negative value.

Positive entries mean that a higher CAPM beta decile has a higher correlation than the symmetric lower CAPM beta decile:

$$\hat{\rho}'_{i,M} < \hat{\rho}'_{10\ \ (i-1),M} \qquad \text{for } 1 \le i \le 5 \tag{30.27}$$

With this we understand why the relationship between the correlations of symmetric deciles appear as they do.

By following similar reasoning, it can be said that to a lesser degree, there is a tendency for consecutive higher deciles to have increasing correlation:

$$\hat{\rho}'_{i,M} < \hat{\rho}'_{i+1,M} \qquad \text{for } 1 \le i \le 9 \tag{31.4}$$

Since by definition the CAPM betas of the deciles have this order:

$$\hat{\beta}'_1 < \hat{\beta}'_2 < \cdots < \hat{\beta}'_9 < \hat{\beta}'_{10} \tag{30.18}$$

And therefore, the reciprocals of the CAPM betas will have this order:

$$\frac{1}{\hat{\beta}'_1} > \frac{1}{\hat{\beta}'_2} > \cdots > \frac{1}{\hat{\beta}'_9} > \frac{1}{\hat{\beta}'_{10}} \tag{30.19}$$

Then there should be a tendency for the absolute value of the residuals divided by their respective CAPM betas to have this relationship:

$$\frac{|\hat{\varepsilon}'_{i,t}|}{\hat{\beta}'_i} > \frac{|\hat{\varepsilon}'_{i+1,t}|}{\hat{\beta}'_{i+1}} \qquad \text{for } 1 \le i \le 9 \tag{31.5}$$

and

$$\sum\left(\frac{\hat{\varepsilon}'_{i,t}}{\beta'_i}\right)^2 > \sum\left(\frac{\hat{\varepsilon}'_{i+1,t}}{\beta'_{i+1}}\right)^2 \tag{31.6}$$

(Compare to lines (30.12) and (30.22).)

Given line (31.6), it might be expected for the correlations of portfolios grouped by CAPM beta deciles to be strictly increasing with CAPM beta. That is, all things being equal, higher CAPM beta deciles are expected to have a higher correlation than the next lower decile, and this is expected to happen monotonically. Additionally, there should be a tendency for this pattern to be true even for individual securities.

However, when the nature of random variation in data is taken into account, all things are not equal. For each of the 8 middle deciles, the range of CAPM betas within the deciles is narrow, as is the range of idiosyncratic risk between deciles.

However, for the 1st and 10th deciles, the range of CAPM betas is broader because those deciles catch the extreme CAPM beta estimates, along with extreme idiosyncratic risk estimates. These facts are reflected in the correlation terms of those deciles. In other words, correlation is expected to be positively correlated with CAPM beta, with a stronger association in the middle 8 deciles since most of the extreme observations have been filtered into the 1st and 10th deciles.

Let me take a moment to explain this. It is economically difficult to generate a CAPM beta that is less than zero, which would be in the first decile. And if less than zero, not by much.

For a CAPM beta to be less than zero, then the economics of the underlying company must be such that when the economy is good the stock goes down, and when the economy is bad, the stock goes up. This is unlikely. More likely, a CAPM beta less than zero would be the result of a statistical anomaly, outliers in the data.

On the other hand, it is easier to generate a CAPM beta greater than 2, which would be in the 10th decile. A CAPM beta above 2 can come from both outliers in the data, and from companies whose underlying price dynamics are highly sensitive to market swings.

Zero and 2 are equidistant from the average CAPM beta of 1.

The cross section of estimated CAPM betas is not distributed normally, uniformly, or even symmetrically. The distribution is skewed right.

To make a less obvious point, if the highest CAPM betas are “really high,” the average CAPM beta remains 1. Which means the low CAPM betas must be low.

I don’t want to get lost in the details, however, the low CAPM betas are not simply generically lower, they cluster around 0. If a market index has a large number of stocks with extremely high CAPM betas, then the clustering will increase. If those high CAPM beta securities are removed from the data for some reason, such as a data integrity issue, counterintuitively, the CAPM betas less than zero will move further away from zero in the negative direction.

Furthermore, since the average CAPM beta will still be 1, then the remaining betas must have moved up to maintain the average. The point is that the 1st and 10th deciles collect the extreme observations of both CAPM beta and idiosyncratic risk.

Looking at Table 31.1 again, we can see that the 10th decile has the largest idiosyncratic risk, and the 1st decile usually has the second highest idiosyncratic risk.

Table 31.1
Idiosyncratic Risk by Decile (31.1)
from *The Volatility Effect: Lower Risk Without Lower Return*
by David Blitz and Pim Van Vliet (2007)

	1	2	3	4	5	6	7	8	9	10
US Market	0.00848	0.00444	0.00405	0.00365	0.00306	0.00150	0.00257	0.00350	0.01007	0.04162
European Market	0.00283	0.00339	0.00221	0.00207	0.00252	0.00227	0.00169	0.00239	0.00521	0.01438
Japanese Market	0.00590	0.00428	0.00343	0.00212	0.00183	0.00164	0.00189	0.00288	0.00576	0.01569

Statistics from *The Capital Asset Pricing Model: Some Empirical Tests*
by Fischer Black, Michael C. Jensen, and Myron Scholes (1972)

	1	2	3	4	5	6	7	8	9	10
Subperiod 1	0.00132	0.00074	0.00046	0.00047	0.00072	0.00031	0.00042	0.00081	0.00100	0.00285
Subperiod 2	0.00030	0.00022	0.00015	0.00013	0.00010	0.00016	0.00017	0.00020	0.00029	0.00263
Subperiod 3	0.00010	0.00008	0.00005	0.00005	0.00005	0.00004	0.00005	0.00006	0.00011	0.00024
Subperiod 4	0.00012	0.00012	0.00007	0.00005	0.00006	0.00005	0.00007	0.00006	0.00010	0.00023

When we look at Table 31.2 again, which is the idiosyncratic risks divided by their respective CAPM betas squared, we see that the first decile is always highest, with the 10th decile almost always being second highest.

Table 31.2
Idiosyncratic Risk Divided by CAPM Beta Squared by Decile (31.2)
from *The Volatility Effect: Lower Risk Without Lower Return*
by David Blitz and Pim Van Vliet (2007)

	1	2	3	4	5	6	7	8	9	10
US Market	0.04187	0.00906	0.00603	0.00494	0.00347	0.00166	0.00252	0.00260	0.00521	0.01328
European Market	0.00691	0.00620	0.00328	0.00240	0.00286	0.00237	0.00150	0.00190	0.00313	0.00648
Japanese Market	0.01587	0.00703	0.00453	0.00256	0.00191	0.00157	0.00168	0.00218	0.00394	0.00778

Statistics from *The Capital Asset Pricing Model: Some Empirical Tests*
by Fischer Black, Michael C. Jensen, and Myron Scholes (1972)

	1	2	3	4	5	6	7	8	9	10
Subperiod 1	0.00562	0.00190	0.00082	0.00064	0.00085	0.00027	0.00030	0.00051	0.00051	0.00120
Subperiod 2	0.00096	0.00051	0.00025	0.00020	0.00012	0.00017	0.00014	0.00014	0.00017	0.00089
Subperiod 3	0.00042	0.00018	0.00009	0.00005	0.00005	0.00003	0.00004	0.00004	0.00006	0.00010
Subperiod 4	0.00032	0.00026	0.00011	0.00007	0.00007	0.00005	0.00006	0.00004	0.00006	0.00011

If line (31.3) is modified to deal with consecutive deciles rather than symmetric deciles we should get this inequality as a tendency:

$$\left(\frac{\hat{\sigma}^2_{\varepsilon\,(i)}}{\hat{\beta}'_i}\right)^2 - \left(\frac{\hat{\sigma}'_{\varepsilon\,(i+1)}}{\hat{\beta}'_{(i+1)}}\right)^2 > 0 \qquad \text{for } 1 \le i \le 9 \qquad (31.7)$$

Here is corresponding Table 31.4:

Table 31.4
Difference in Idiosyncratic Risk Divided by CAPM Beta Squared by Sequential Decile (31.6)
Statistics from *The Volatility Effect: Lower Risk Without Lower Return*
by David Blitz and Pim Van Vliet (2007)

	1st-2nd	2nd-3rd	3rd-4th	4th-5th	5th-6th	6th-7th	7th-8th	8th-9th	9th-10th
US Market	0.03281	0.00303	0.00109	0.00147	0.00181	-0.00086	-0.00008	-0.00261	-0.00807
European Market	0.00072	0.00291	0.00089	-0.00046	0.00049	0.00086	-0.00040	-0.00122	-0.00335
Japanese Market	0.00884	0.00250	0.00197	0.00066	0.00033	-0.00011	-0.00050	-0.00176	-0.00385

Statistics from *The Capital Asset Pricing Model: Some Empirical Tests*
by Fischer Black, Michael C. Jensen, and Myron Scholes (1972)

	1st-2nd	2nd-3rd	3rd-4th	4th-5th	5th-6th	6th-7th	7th-8th	8th-9th	9th-10th
Subperiod 1	0.003718	0.001084	0.000179	-0.000213	0.000585	-0.000038	-0.000207	0.000002	-0.000689
Subperiod 2	0.000453	0.000259	0.000051	0.000081	-0.000052	0.000026	0.000004	-0.000030	-0.000725
Subperiod 3	0.000246	0.000090	0.000032	0.000006	0.000014	-0.000005	-0.000004	-0.000019	-0.000039
Subperiod 4	0.000058	0.000154	0.000037	0.000005	0.000016	-0.000012	0.000019	-0.000018	-0.000047

Each negative entry is a place where correlation is not increasing from one decile to the next higher decile. The multiple negative entries seen in the higher consecutive deciles is not what I would expect. The asymmetric distribution of CAPM betas in the cross-section likely has something to do with it. There is more work to be done on this point.

Note that in each case the value corresponding to the 2nd decile is positive with respect to the 1st decile, but also in each case the 10th decile is negative with respect to the 9th decile. This means, as expected, in every case the 2nd decile has a higher correlation than the first, but also, as expected, in every case the 10th decile has a lower correlation than the 9th decile.

Finally, we need to check to see if CAPM beta is correlated with correlation. Here are the correlations between CAPM beta and correlation for all 10 deciles and for the middle eight deciles.

Table 31.5
Correlation Between CAPM β' and Correlation
Statistics from *The Volatility Effect: Lower Risk Without Lower Return* by David Blitz and Pim Van Vliet (2007)

	All 10 Deciles	Middle 8 Deciles
US	0.3131	0.4126
Europe	0.1540	0.5751
Japan	0.4231	0.5472

Statistics from *The Capital Asset Pricing Model: Some Empirical Tests* by Fischer Black, Michael C. Jensen, and Myron Scholes (1972)

	All 10 Deciles	Middle 8 Deciles
Subperiod 1	0.6171	0.7414
Subperiod 2	-0.0190	0.6680
Subperiod 3	0.6406	0.7038
Subperiod 4	0.6484	0.7421

Notice the positive correlation for every grouping of 10 deciles except in the second time period of BJS. As predicted, there is a stronger correlation for the middle eight deciles.

By understanding the structure of the relationship between correlation calculations for each decile as derived in this and the last video, the strong correlation of the middle eight deciles is not surprising, but is expected.

Lastly, looking at the data from 2015 - 2019 as used in video 21, the correlation between the CAPM beta of each security and its respective correlation is 0.82, which I think is very high.